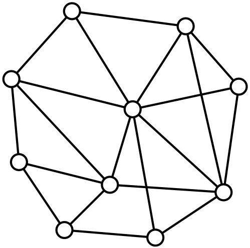
Gallai's theorem on colour-critical graphs and related results

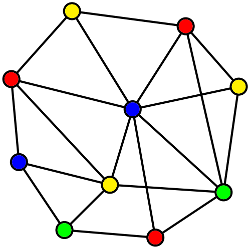
Matěj Stehlík

September 18, 2009



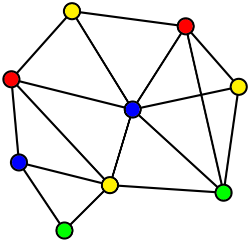
- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.

Chromatic number and critical graphs

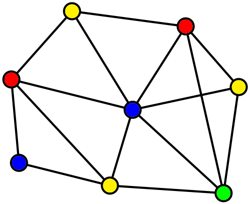


- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.

Chromatic number and critical graphs

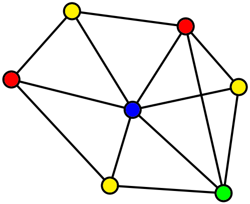


- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.



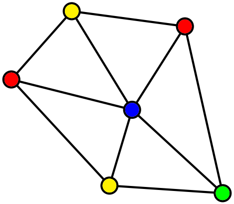
- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.

Chromatic number and critical graphs



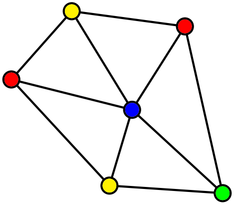
- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.

Chromatic number and critical graphs



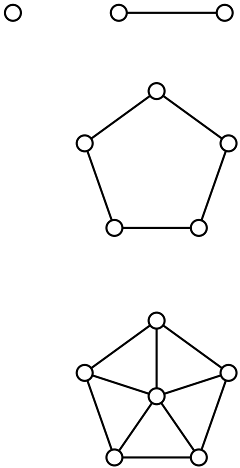
- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.

Chromatic number and critical graphs



- A colouring of a graph G is an assignment of colours to the vertices of G so that adjacent vertices receive different colours.
- The chromatic number $\chi(G)$ is the minimum number of colours required to colour the graph G.
- A graph G is k-critical if $\chi(G) = k$ and $\chi(G - x) < k$, for every $x \in V(G)$.

Basic facts about critical graphs



- The only 1-critical graph is K_1
- The only 2-critical graph is K_2
- The only 3-critical graphs are the odd cycles C_{2n-1}
- No good characterisation of k-critical graphs for $k \geq 4$ (determining whether a graph is k-colourable is NP-hard for $k \geq 3$)
- Every vertex in a k-critical graph has degree at least $k - 1$

Gallai's Theorem

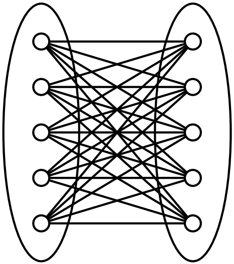
Gallai's Theorem (1963)

If G is a k-critical graph with a connected complement, then

$$|V(G)| \geqslant 2k - 1.$$

- So a k-critical graph G with less than $2k - 1$ vertices is the complete join of a k_1-critical graph G_1 and k_2-critical graph G_2, where $k_1 + k_2 = k$.
- W_5 is 4-critical and $|V(W_5)| = 6$, so by Gallai's theorem it is the complete join of two smaller critical graphs G_1 and G_2.

Gallai's Theorem



G_1 G_2

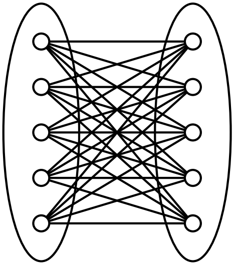
Gallai's Theorem (1963)

If G is a k-critical graph with a connected complement, then

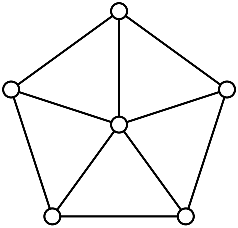
$$|V(G)| \geqslant 2k - 1.$$

- So a k-critical graph G with less than $2k - 1$ vertices is the complete join of a k_1-critical graph G_1 and k_2-critical graph G_2, where $k_1 + k_2 = k$.
- W_5 is 4-critical and $|V(W_5)| = 6$, so by Gallai's theorem it is the complete join of two smaller critical graphs G_1 and G_2.

Gallai's Theorem



G_1 G_2



W_5

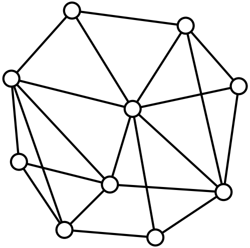
Gallai's Theorem (1963)

If G is a k-critical graph with a connected complement, then

$$|V(G)| \geqslant 2k - 1.$$

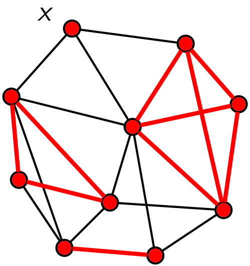
- So a k-critical graph G with less than $2k - 1$ vertices is the complete join of a k_1-critical graph G_1 and k_2-critical graph G_2, where $k_1 + k_2 = k$.
- W_5 is 4-critical and $|V(W_5)| = 6$, so by Gallai's theorem it is the complete join of two smaller critical graphs G_1 and G_2.

Colouring the complement



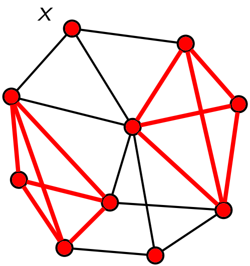
- A *k-cover* T of G is a partition of G into k disjoint cliques.
- The minimum k for which G has a k-cover is the *cover number* $\overline{\chi}(G)$.
- A vertex $x \in V(G)$ is *cover-critical* if $\overline{\chi}(G - x) = \overline{\chi}(G) - 1$.
- Given a cover-critical vertex x of G, a cover T of G is *x-extreme* if $T - x$ is a minimal cover of $G - x$ with the minimum number of isolated vertices.
- A k-cover of G corresponds to a k-colouring of $\overline{G}$, and $\overline{\chi}(G) = \chi(\overline{G})$.

Colouring the complement



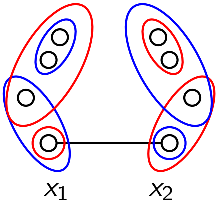
- A k-cover T of G is a partition of G into k disjoint cliques.
- The minimum k for which G has a k-cover is the cover number $\overline{\chi}(G)$.
- A vertex $x \in V(G)$ is cover-critical if $\overline{\chi}(G - x) = \overline{\chi}(G) - 1$.
- Given a cover-critical vertex x of G, a cover T of G is x-extreme if $T - x$ is a minimal cover of $G - x$ with the minimum number of isolated vertices.
- A k-cover of G corresponds to a k-colouring of $\overline{G}$, and $\overline{\chi}(G) = \chi(\overline{G})$.

Colouring the complement



- A *k-cover* T of G is a partition of G into k disjoint cliques.
- The minimum k for which G has a k-cover is the cover number $\overline{\chi}(G)$.
- A vertex $x \in V(G)$ is cover-critical if $\overline{\chi}(G - x) = \overline{\chi}(G) - 1$.
- Given a cover-critical vertex x of G, a cover T of G is x-extreme if $T - x$ is a minimal cover of $G - x$ with the minimum number of isolated vertices.
- A k-cover of G corresponds to a k-colouring of $\overline{G}$, and $\overline{\chi}(G) = \chi(\overline{G})$.

A simple observation

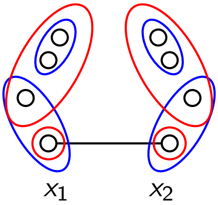


Let $I(T)$ be the set of isolated vertices of T.

Lemma

Let x_1x_2 be an edge and let T_1 and T_2 be minimal covers s.t. $x_1 \in I(T_1)$ and $x_2 \in I(T_2)$. Then x_1 and x_2 lie in the same component of $T_1 \cup T_2$.

A simple observation

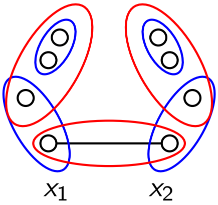


Let $I(T)$ be the set of isolated vertices of T.

Lemma

Let x_1x_2 be an edge and let T_1 and T_2 be minimal covers s.t. $x_1 \in I(T_1)$ and $x_2 \in I(T_2)$. Then x_1 and x_2 lie in the same component of $T_1 \cup T_2$.

A simple observation

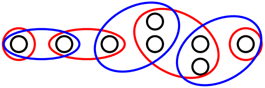


Let $I(T)$ be the set of isolated vertices of T.

Lemma

Let x_1x_2 be an edge and let T_1 and T_2 be minimal covers s.t. $x_1 \in I(T_1)$ and $x_2 \in I(T_2)$. Then x_1 and x_2 lie in the same component of $T_1 \cup T_2$.

The key lemmas



Lemma

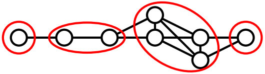
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If $x_1 x_2$ is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

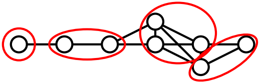
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If $x_1 x_2$ is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

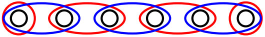
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If x_1x_2 is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

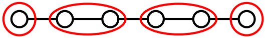
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If x_1x_2 is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

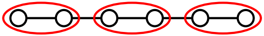
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If x_1x_2 is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

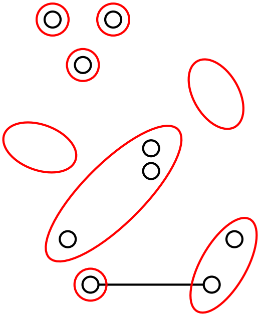
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If x_1x_2 is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

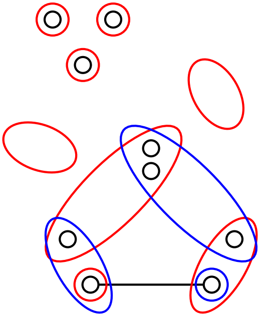
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If x_1x_2 is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

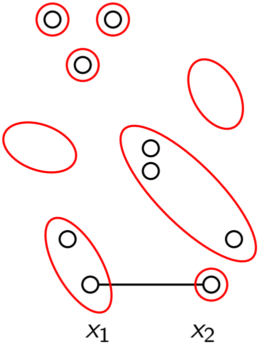
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If x_1x_2 is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

$$I(T_1 - x_1) = I(T_2 - x_2).$$

The key lemmas



Lemma

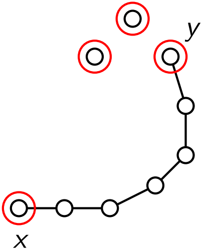
If T_1 is an x-extreme cover and T_2 is any cover, then any component of $T_1 \cup T_2$ contains at most one isolated vertex of T_1.

Lemma

If $x_1 x_2$ is an edge and T_1 is an x_1-extreme cover, then there exists an x_2-extreme cover T_2 s.t.

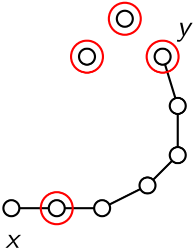
$$I(T_1 - x_1) = I(T_2 - x_2).$$

The final step



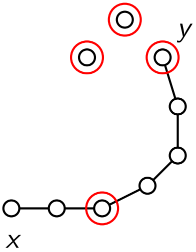
- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

The final step



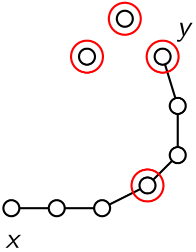
- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

The final step



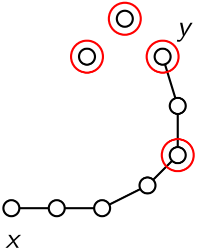
- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

The final step



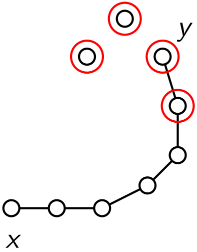
- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

The final step



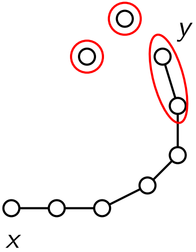
- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

The final step



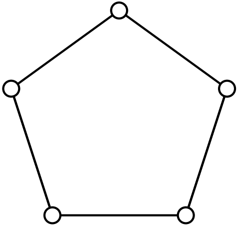
- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

The final step



- Suppose T is an x-extreme cover containing another isolated vertex $y \neq x$.
- Use the previous lemma to shift the isolated vertex from x to a neighbour of y.
- Now we can obtain a smaller cover; a contradiction.
- Hence no x-extreme cover contains another isolated vertex $y \neq x$.
- Hence $|V(G)| \geq 2\overline{\chi}(G) - 1$.

Matchings and factor-critical graphs



- A matching of G is a set of pairwise non-adjacent edges of G.
- The maximum size of a matching of G is the matching number $\nu(G)$.
- A graph G is factor-critical if $G - x$ has a perfect matching for every $x \in V(G)$.
- If G is a factor-critical graph then $\nu(G - x) = \nu(G)$ for every $x \in V(G)$.

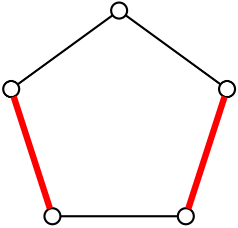
Gallai's Lemma (1963)

If G is a connected graph such that

$$\nu(G - x) = \nu(G)$$

for every $x \in V(G)$, then G is factor-critical.

Matchings and factor-critical graphs



- A matching of G is a set of pairwise non-adjacent edges of G.
- The maximum size of a matching of G is the matching number $\nu(G)$.
- A graph G is factor-critical if $G - x$ has a perfect matching for every $x \in V(G)$.
- If G is a factor-critical graph then $\nu(G - x) = \nu(G)$ for every $x \in V(G)$.

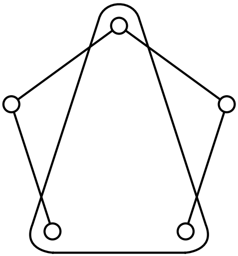
Gallai's Lemma (1963)

If G is a connected graph such that

$$\nu(G - x) = \nu(G)$$

for every $x \in V(G)$, then G is factor-critical.

Expansion number and t-stable hypergraphs



- The expansion number $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is

$$t(\mathscr{H}) := \max_{\mathscr{H}' \subseteq \mathscr{H}} \{|V(\mathscr{H}')| - |\mathscr{H}'|\}.$$

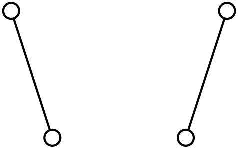
- A hypergraph $\mathscr{H}$ is t-stable if $t(\mathscr{H} - x) = t(\mathscr{H})$ for every $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$.

Füredi's Theorem (1986)

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

Expansion number and t-stable hypergraphs



- The expansion number $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is

$$t(\mathscr{H}) := \max_{\mathscr{H}' \subseteq \mathscr{H}} \{|V(\mathscr{H}')| - |\mathscr{H}'|\}.$$

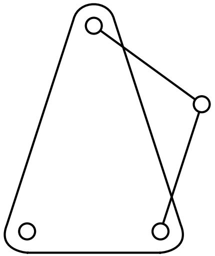
- A hypergraph $\mathscr{H}$ is t-stable if $t(\mathscr{H} - x) = t(\mathscr{H})$ for every $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$.

Füredi's Theorem (1986)

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

Expansion number and t-stable hypergraphs



- The expansion number $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is

$$t(\mathscr{H}) := \max_{\mathscr{H}' \subseteq \mathscr{H}} \{|V(\mathscr{H}')| - |\mathscr{H}'|\}.$$

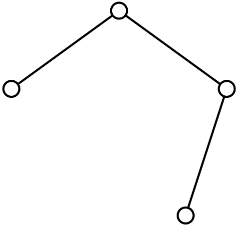
- A hypergraph $\mathscr{H}$ is t-stable if $t(\mathscr{H} - x) = t(\mathscr{H})$ for every $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$.

Füredi's Theorem (1986)

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

Expansion number and t-stable hypergraphs

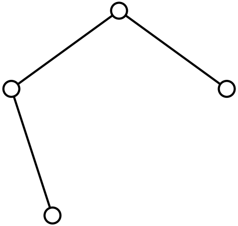


- The expansion number $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is
$$t(\mathscr{H}) := \max_{\mathscr{H}' \subseteq \mathscr{H}} \{|V(\mathscr{H}')| - |\mathscr{H}'|\}.$$
- A hypergraph $\mathscr{H}$ is t-stable if $t(\mathscr{H} - x) = t(\mathscr{H})$ for every $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$.

Füredi's Theorem (1986)

If $\mathscr{H}$ is a connected t-stable hypergraph, then
$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

Expansion number and t-stable hypergraphs



- The expansion number $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is

$$t(\mathscr{H}) := \max_{\mathscr{H}' \subseteq \mathscr{H}} \{|V(\mathscr{H}')| - |\mathscr{H}'|\}.$$

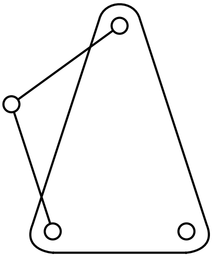
- A hypergraph $\mathscr{H}$ is t-stable if $t(\mathscr{H} - x) = t(\mathscr{H})$ for every $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$.

Füredi's Theorem (1986)

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

Expansion number and t-stable hypergraphs



- The expansion number $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is

$$t(\mathscr{H}) := \max_{\mathscr{H}' \subseteq \mathscr{H}} \{|V(\mathscr{H}')| - |\mathscr{H}'|\}.$$

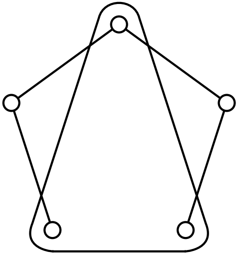
- A hypergraph $\mathscr{H}$ is t-stable if $t(\mathscr{H} - x) = t(\mathscr{H})$ for every $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$.

Füredi's Theorem (1986)

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

Transversals and τ-critical hypergraphs



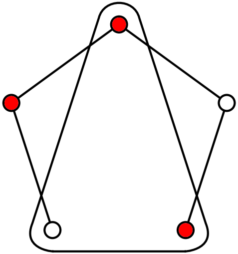
- A transversal (vertex cover, blocking set) of a hypergraph $\mathscr{H}$ is a subset $T \subseteq V(\mathscr{H})$ which meets every edge of $\mathscr{H}$.
- The minimal size of a transversal is the transversal number $\tau(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is τ-critical if $\tau(\mathscr{H} - E) < \tau(\mathscr{H})$, for every $E \in \mathscr{H}$.

Theorem (MS 2006)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

Transversals and τ-critical hypergraphs



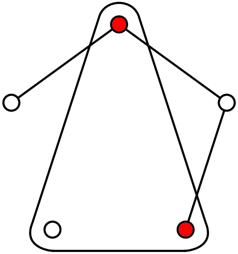
- A transversal (vertex cover, blocking set) of a hypergraph $\mathscr{H}$ is a subset $T \subseteq V(\mathscr{H})$ which meets every edge of $\mathscr{H}$.
- The minimal size of a transversal is the transversal number $\tau(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is τ-critical if $\tau(\mathscr{H} - E) < \tau(\mathscr{H})$, for every $E \in \mathscr{H}$.

Theorem (MS 2006)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

Transversals and τ-critical hypergraphs



- A transversal (vertex cover, blocking set) of a hypergraph $\mathscr{H}$ is a subset $T \subseteq V(\mathscr{H})$ which meets every edge of $\mathscr{H}$.
- The minimal size of a transversal is the transversal number $\tau(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is τ-critical if $\tau(\mathscr{H} - E) < \tau(\mathscr{H})$, for every $E \in \mathscr{H}$.

Theorem (MS 2006)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

Implications between the theorems

Füredi's Theorem

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

$\Longleftrightarrow$

Theorem (MS 06)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

Füredi's Theorem $\Downarrow$

Gallai's Lemma

If G is a connected graph such that $\nu(G - x) = \nu(G)$ for every $x \in V(G)$, then G is factor-critical.

Theorem (MS 06) $\Downarrow$

Gallai's Theorem

If G is a χ-critical graph with a connected complement, then

$$|V(G)| \geq 2\chi(G) - 1.$$

Implications between the theorems

Füredi's Theorem

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

$\Longleftrightarrow$

Theorem (MS 06)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

$\Downarrow$ $\Downarrow$

Gallai's Lemma

If G is a connected graph such that $\nu(G - x) = \nu(G)$ for every $x \in V(G)$, then G is factor-critical.

Gallai's Theorem

If G is a χ-critical graph with a connected complement, then

$$|V(G)| \geq 2\chi(G) - 1.$$

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then

$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

A link between t and ν

Lemma

If G is a graph, then
$$t(G) = \nu(G).$$

- If M is a max. matching of G, then $|V(M)| - |M| = \nu(G)$.
- Hence $t(G) \geq \nu(G)$.
- Let $H \subseteq G$ be a subgraph s.t. $|V(H)| - |H| = t(G)$.
- If some vertex x has deg > 1, delete an edge e incident with x.
- $|V(H - e)| \geq |V(H)| - 1$ and $|H - e| = |H| - 1$
- Hence $|V(H - e)| - |H - e| \geq |V(H)| - |H|$
- Keep deleting edges until every vertex has deg ≤ 1.
- We end up with a matching M such that $|V(M)| - |M| \geq |V(H)| - |H| = t(G)$.
- Hence $\nu(G) \geq t(G)$.

How Füredi's Theorem implies Gallai's Lemma

- Suppose G is a connected ν-stable graph.
- G is connected and t-stable by previous lemma.
- $|V(G)| \leq 2t(G) + 1 = 2\nu(G) + 1$ by Füredi's Theorem.
- So for any vertex x of G,

$$\begin{aligned} |V(G-x)| &= |V(G)| - 1 \\ &\leq (2\nu(G) + 1) - 1 \\ &= 2\nu(G-x). \end{aligned}$$

- So $G - x$ has a perfect matching.
- Hence G is factor-critical.

How Füredi's Theorem implies Gallai's Lemma

- Suppose G is a connected ν-stable graph.
- G is connected and t-stable by previous lemma.
- $|V(G)| \leq 2t(G) + 1 = 2\nu(G) + 1$ by Füredi's Theorem.
- So for any vertex x of G,

$$\begin{aligned} |V(G-x)| &= |V(G)| - 1 \\ &\leq (2\nu(G) + 1) - 1 \\ &= 2\nu(G-x). \end{aligned}$$

- So $G - x$ has a perfect matching.
- Hence G is factor-critical.

How Füredi's Theorem implies Gallai's Lemma

- Suppose G is a connected ν-stable graph.
- G is connected and t-stable by previous lemma.
- $|V(G)| \leq 2t(G) + 1 = 2\nu(G) + 1$ by Füredi's Theorem.
- So for any vertex x of G,

$$\begin{aligned} |V(G-x)| &= |V(G)| - 1 \\ &\leq (2\nu(G)+1) - 1 \\ &= 2\nu(G-x). \end{aligned}$$

- So $G - x$ has a perfect matching.
- Hence G is factor-critical.

How Füredi's Theorem implies Gallai's Lemma

- Suppose G is a connected ν-stable graph.
- G is connected and t-stable by previous lemma.
- $|V(G)| \leq 2t(G) + 1 = 2\nu(G) + 1$ by Füredi's Theorem.
- So for any vertex x of G,

$$\begin{aligned} |V(G-x)| &= |V(G)| - 1 \\ &\leq (2\nu(G)+1) - 1 \\ &= 2\nu(G-x). \end{aligned}$$

- So $G - x$ has a perfect matching.
- Hence G is factor-critical.

How Füredi's Theorem implies Gallai's Lemma

- Suppose G is a connected ν-stable graph.
- G is connected and t-stable by previous lemma.
- $|V(G)| \leq 2t(G) + 1 = 2\nu(G) + 1$ by Füredi's Theorem.
- So for any vertex x of G,

$$\begin{aligned} |V(G - x)| &= |V(G)| - 1 \\ &\leq (2\nu(G) + 1) - 1 \\ &= 2\nu(G - x). \end{aligned}$$

- So $G - x$ has a perfect matching.
- Hence G is factor-critical.

How Füredi's Theorem implies Gallai's Lemma

- Suppose G is a connected ν-stable graph.
- G is connected and t-stable by previous lemma.
- $|V(G)| \leq 2t(G) + 1 = 2\nu(G) + 1$ by Füredi's Theorem.
- So for any vertex x of G,

$$\begin{aligned} |V(G - x)| &= |V(G)| - 1 \\ &\leq (2\nu(G) + 1) - 1 \\ &= 2\nu(G - x). \end{aligned}$$

- So $G - x$ has a perfect matching.
- Hence G is factor-critical.

Implications between the theorems

Füredi's Theorem

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \leq 2t(\mathscr{H}) + 1.$$

$\Leftrightarrow$

Theorem (MS 06)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

$\Downarrow$ $\Downarrow$

Gallai's Lemma

If G is a connected graph such that $\nu(G - x) = \nu(G)$ for every $x \in V(G)$, then G is factor-critical.

Gallai's Theorem

If G is a χ-critical graph with a connected complement, then

$$|V(G)| \geq 2\chi(G) - 1.$$

A link between t and τ

Lemma

$t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$

- Pick $\mathscr{H}' \subseteq \mathscr{H}$ s.t. $|V(\mathscr{H}')| - |\mathscr{H}'| = t(\mathscr{H})$.
- If a vertex x of $\mathscr{H}$ is not in $\mathscr{H}'$ add an edge E containing x.
- Repeat until we obtain an edge cover $\mathscr{H}''$ of $\mathscr{H}$.
- Note that $|V(\mathscr{H}'')| - |\mathscr{H}''| \geq |V(\mathscr{H}')| - |\mathscr{H}'|$, so

$$\begin{aligned} t(\mathscr{H}) &= |V(\mathscr{H}')| - |\mathscr{H}'| \\ &\leq |V(\mathscr{H}'')| - |\mathscr{H}''| \\ &\leq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

- By considering a minimum edge cover of $\mathscr{H}$,

$$\begin{aligned} t(\mathscr{H}) &\geq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

A link between t and τ

Lemma

$t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$

- Pick $\mathscr{H}' \subseteq \mathscr{H}$ s.t. $|V(\mathscr{H}')| - |\mathscr{H}'| = t(\mathscr{H})$.
- If a vertex x of $\mathscr{H}$ is not in $\mathscr{H}'$ add an edge E containing x.
- Repeat until we obtain an edge cover $\mathscr{H}''$ of $\mathscr{H}$.
- Note that $|V(\mathscr{H}'')| - |\mathscr{H}''| \geq |V(\mathscr{H}')| - |\mathscr{H}'|$, so

$$\begin{aligned} t(\mathscr{H}) &= |V(\mathscr{H}')| - |\mathscr{H}'| \\ &\leq |V(\mathscr{H}'')| - |\mathscr{H}''| \\ &\leq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

- By considering a minimum edge cover of $\mathscr{H}$,

$$\begin{aligned} t(\mathscr{H}) &\geq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

A link between t and τ

Lemma

$t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$

- Pick $\mathscr{H}' \subseteq \mathscr{H}$ s.t. $|V(\mathscr{H}')| - |\mathscr{H}'| = t(\mathscr{H})$.
- If a vertex x of $\mathscr{H}$ is not in $\mathscr{H}'$ add an edge E containing x.
- Repeat until we obtain an edge cover $\mathscr{H}''$ of $\mathscr{H}$.
- Note that $|V(\mathscr{H}'')| - |\mathscr{H}''| \geq |V(\mathscr{H}')| - |\mathscr{H}'|$, so

$$\begin{aligned} t(\mathscr{H}) &= |V(\mathscr{H}')| - |\mathscr{H}'| \\ &\leq |V(\mathscr{H}'')| - |\mathscr{H}''| \\ &\leq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

- By considering a minimum edge cover of $\mathscr{H}$,

$$\begin{aligned} t(\mathscr{H}) &\geq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

A link between t and τ

Lemma

$t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$

- Pick $\mathscr{H}' \subseteq \mathscr{H}$ s.t. $|V(\mathscr{H}')| - |\mathscr{H}'| = t(\mathscr{H})$.
- If a vertex x of $\mathscr{H}$ is not in $\mathscr{H}'$ add an edge E containing x.
- Repeat until we obtain an edge cover $\mathscr{H}''$ of $\mathscr{H}$.
- Note that $|V(\mathscr{H}'')| - |\mathscr{H}''| \geq |V(\mathscr{H}')| - |\mathscr{H}'|$, so

$$\begin{aligned} t(\mathscr{H}) &= |V(\mathscr{H}')| - |\mathscr{H}'| \\ &\leq |V(\mathscr{H}'')| - |\mathscr{H}''| \\ &\leq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

- By considering a minimum edge cover of $\mathscr{H}$,

$$\begin{aligned} t(\mathscr{H}) &\geq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

A link between t and τ

Lemma

$t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$

- Pick $\mathscr{H}' \subseteq \mathscr{H}$ s.t. $|V(\mathscr{H}')| - |\mathscr{H}'| = t(\mathscr{H})$.
- If a vertex x of $\mathscr{H}$ is not in $\mathscr{H}'$ add an edge E containing x.
- Repeat until we obtain an edge cover $\mathscr{H}''$ of $\mathscr{H}$.
- Note that $|V(\mathscr{H}'')| - |\mathscr{H}''| \geq |V(\mathscr{H}')| - |\mathscr{H}'|$, so

$$\begin{aligned} t(\mathscr{H}) &= |V(\mathscr{H}')| - |\mathscr{H}'| \\ &\leq |V(\mathscr{H}'')| - |\mathscr{H}''| \\ &\leq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

- By considering a minimum edge cover of $\mathscr{H}$,

$$\begin{aligned} t(\mathscr{H}) &\geq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

A link between t and τ

Lemma

$t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$

- Pick $\mathscr{H}' \subseteq \mathscr{H}$ s.t. $|V(\mathscr{H}')| - |\mathscr{H}'| = t(\mathscr{H})$.
- If a vertex x of $\mathscr{H}$ is not in $\mathscr{H}'$ add an edge E containing x.
- Repeat until we obtain an edge cover $\mathscr{H}''$ of $\mathscr{H}$.
- Note that $|V(\mathscr{H}'')| - |\mathscr{H}''| \geq |V(\mathscr{H}')| - |\mathscr{H}'|$, so

$$\begin{aligned} t(\mathscr{H}) &= |V(\mathscr{H}')| - |\mathscr{H}'| \\ &\leq |V(\mathscr{H}'')| - |\mathscr{H}''| \\ &\leq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

- By considering a minimum edge cover of $\mathscr{H}$,

$$\begin{aligned} t(\mathscr{H}) &\geq |V(\mathscr{H})| - \rho(\mathscr{H}) \\ &= |\mathscr{H}^*| - \tau(\mathscr{H}^*) \end{aligned}$$

Equivalence of Füredi's Theorem and Theorem (MS 06)

- Note that

$$\begin{aligned} |V(\mathscr{H})| &\leq 2t(\mathscr{H}) + 1 \\ &\leq 2(|\mathscr{H}^*| - \tau(\mathscr{H}^*)) + 1 \\ 2\tau(\mathscr{H}^*) - 1 &\leq |\mathscr{H}^*|. \end{aligned}$$

- To show that $\mathscr{H}$ is t-stable iff $\mathscr{H}^*$ is τ-critical is slightly technical...

Equivalence of Füredi's Theorem and Theorem (MS 06)

- Note that

$$\begin{aligned} |V(\mathscr{H})| &\leq 2t(\mathscr{H}) + 1 \\ &\leq 2(|\mathscr{H}^*| - \tau(\mathscr{H}^*)) + 1 \\ 2\tau(\mathscr{H}^*) - 1 &\leq |\mathscr{H}^*|. \end{aligned}$$

- To show that $\mathscr{H}$ is t-stable iff $\mathscr{H}^*$ is τ-critical is slightly technical...

Implications between the theorems

Füredi's Theorem

If $\mathscr{H}$ is a connected t-stable hypergraph, then

$$|V(\mathscr{H})| \le 2t(\mathscr{H}) + 1.$$

$\Longleftrightarrow$

Theorem (MS 06)

If $\mathscr{H}$ is a connected τ-critical hypergraph, then

$$|\mathscr{H}| \ge 2\tau(\mathscr{H}) - 1.$$

$\Downarrow$ $\Downarrow$

Gallai's Lemma

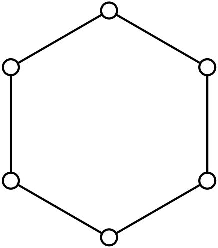
If G is a connected graph such that $\nu(G - x) = \nu(G)$ for every $x \in V(G)$, then G is factor-critical.

Gallai's Theorem

If G is a χ-critical graph with a connected complement, then

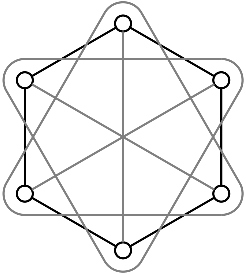
$$|V(G)| \ge 2\chi(G) - 1.$$

A link between χ and τ



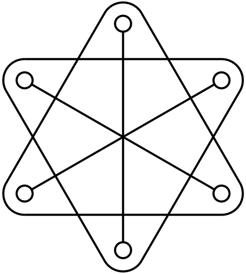
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

A link between χ and τ



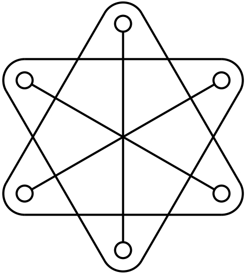
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

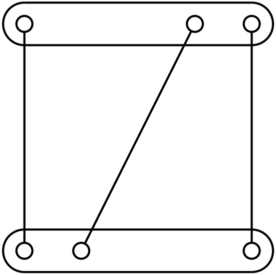
A link between χ and τ



- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

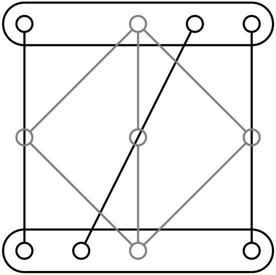
A link between χ and τ





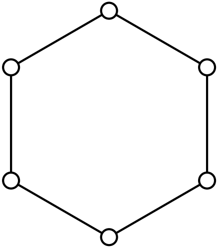
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

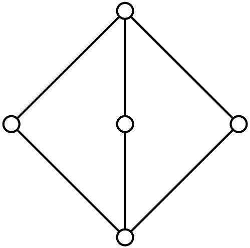
A link between χ and τ



- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

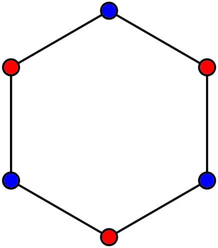
A link between χ and τ





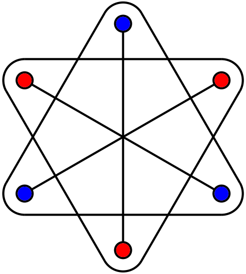
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

A link between χ and τ



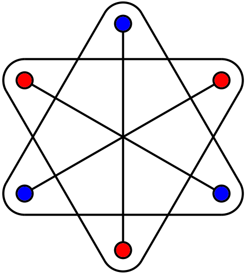
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

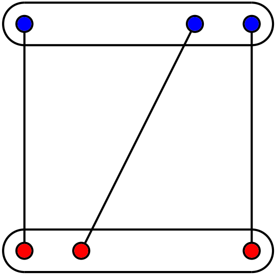
A link between χ and τ



- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

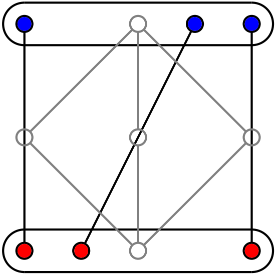
A link between χ and τ





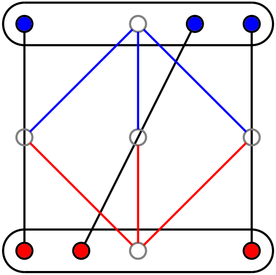
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

A link between χ and τ



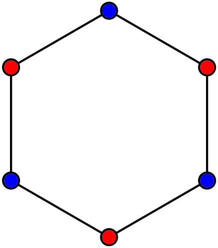
- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

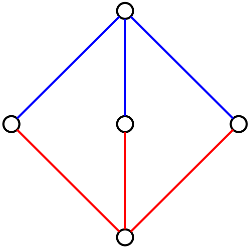
A link between χ and τ



- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

A link between χ and τ





- For any graph G, form the hypergraph $\mathscr{A}(G)$ with the maximal independent sets of G, and let $\mathscr{A}^*(G)$ be the dual of $\mathscr{A}(G)$.
- $\mathscr{A}^*(G)$ has the Helly property: every pairwise intersecting partial hypergraph is a star. ($\mathscr{A}^*(G)$ is 'triangle-free').
- $\overline{G}$ is connected iff $\mathscr{A}^*(G)$ is connected.
- $|V(G)| = |\mathscr{A}^*(G)|$.
- $\chi(G) = \tau(\mathscr{A}^*(G))$.

How Theorem (MS 06) implies Gallai's Theorem

By the previous discussion, Gallai's Theorem may be rephrased as follows:

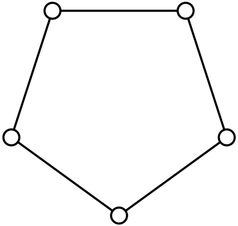
Theorem

If $\mathscr{H}$ is a connected τ-critical Helly hypergraph, then

$$|\mathscr{H}| \geq 2\tau(\mathscr{H}) - 1.$$

So Theorem (MS 06) extends Gallai's Theorem to non-Helly hypergraphs.

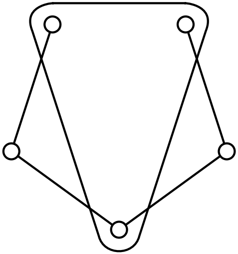
Minimal connected τ-critical hypergraphs



A connected τ-critical hypergraph $\mathscr{H}$ is minimal if $|\mathscr{H}| = 2\tau(\mathscr{H}) - 1$.

Proposition

If $\mathscr{H}$ is a minimal connected τ-critical hypergraph, then $L(\mathscr{H})$ is a factor-critical graph.



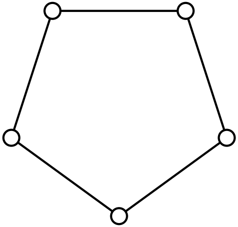
Theorem

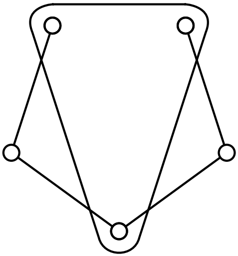
If $\mathscr{H}$ is a minimal connected τ-critical hypergraph, then

$$|V(\mathscr{H})| \geqslant |\mathscr{H}|.$$

If $|V(\mathscr{H})| = |\mathscr{H}|$, $\mathscr{H}$ is square.

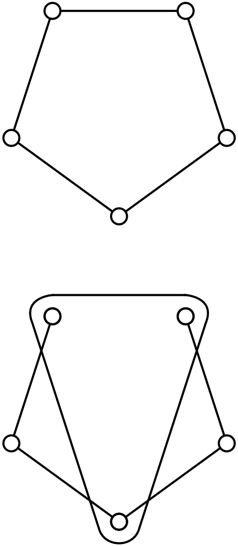
Hypergraph colouring



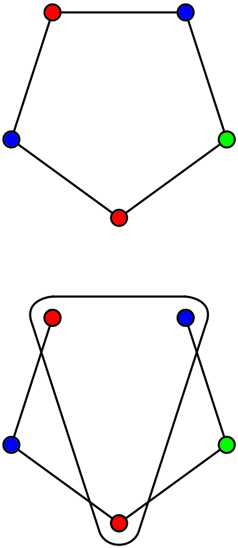


- A colouring of a hypergraph is an assignment of colours to the vertices so that no edge is monochromatic.
- The minimal number of colours in a colouring of $\mathscr{H}$ is the chromatic number $\chi(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is χ-critical if $\chi(\mathscr{H} - E) < \chi(\mathscr{H})$, for every $E \in \mathscr{H}$.

Hypergraph colouring

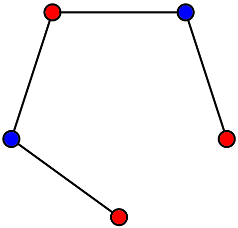


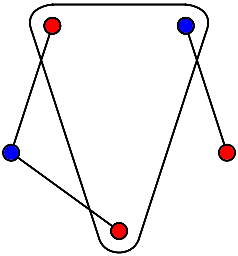
- A colouring of a hypergraph is an assignment of colours to the vertices so that no edge is monochromatic.
- The minimal number of colours in a colouring of $\mathscr{H}$ is the chromatic number $\chi(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is χ-critical if $\chi(\mathscr{H} - E) < \chi(\mathscr{H})$, for every $E \in \mathscr{H}$.



- A colouring of a hypergraph is an assignment of colours to the vertices so that no edge is monochromatic.
- The minimal number of colours in a colouring of $\mathscr{H}$ is the chromatic number $\chi(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is χ-critical if $\chi(\mathscr{H} - E) < \chi(\mathscr{H})$, for every $E \in \mathscr{H}$.

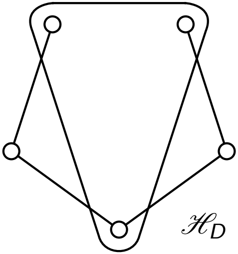
Hypergraph colouring

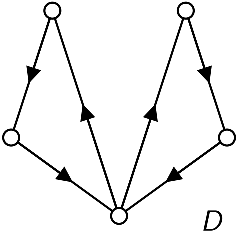




- A colouring of a hypergraph is an assignment of colours to the vertices so that no edge is monochromatic.
- The minimal number of colours in a colouring of $\mathscr{H}$ is the chromatic number $\chi(\mathscr{H})$.
- A hypergraph $\mathscr{H}$ is χ-critical if $\chi(\mathscr{H} - E) < \chi(\mathscr{H})$, for every $E \in \mathscr{H}$.

A characterisation theorem





Let $\mathscr{H}_D = \{\{x\} \cup N^+(x) \mid x \in V(D)\}$.

Theorem (Seymour 1974)

For any square hypergraph $\mathscr{H}$, TFAE:

- *$\mathscr{H}$ is χ-critical 3-chromatic*
- *$\mathscr{H} \cong \mathscr{H}_D$, where D is strongly connected and contains no directed even circuits.*

Theorem

For any square hypergraph $\mathscr{H}$, TFAE:

- *$\mathscr{H}$ is minimal connected τ-critical;*
- *$\mathscr{H}$ is χ-critical 3-chromatic and $|\mathscr{H}| < 2\tau(\mathscr{H})$;*

- Can a Gallai-type theorem be proved for other invariants?
- In particular, what about the list chromatic number?
- Can we prove some structural result for k-critical graphs with a on at most (say) $3k - 3$ vertices? (Seems difficult. . .)

Some open questions

- Can a Gallai-type theorem be proved for other invariants?
- In particular, what about the list chromatic number?
- Can we prove some structural result for k-critical graphs with a on at most (say) $3k - 3$ vertices? (Seems difficult. . .)

Some open questions

- Can a Gallai-type theorem be proved for other invariants?
- In particular, what about the list chromatic number?
- Can we prove some structural result for k-critical graphs with a on at most (say) $3k - 3$ vertices? (Seems difficult...)

Merci !