The Spanning Galaxy Problem

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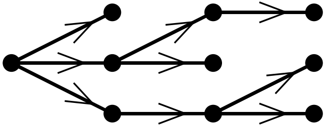
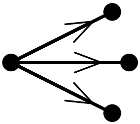
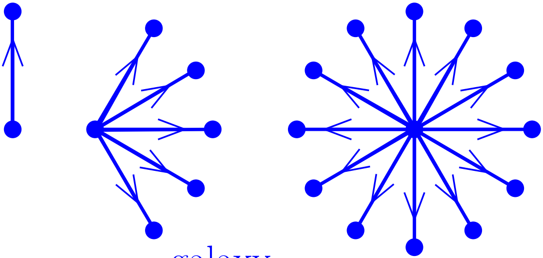
Definitions



arborescence



star



galaxy

Spanning galaxy problem

A subdigraph D' of D is spanning if $V(D') = V(D)$.

Spanning Galaxy Problem:
Instance: A digraph D.
Question: Does D have a spanning galaxy?

Theorem : Spanning Galaxy Problem is NP-complete *even when restricted to digraphs which are* acyclic, planar, bipartite, subcubic, with arbitrary girth, and with maximum outdegree 2.

Arborescences

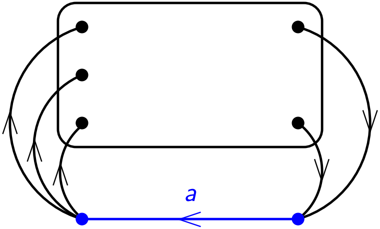
The Spanning Galaxy Problem is solvable in linear time when restricted to arborescences.

Lemma : Every arborescence T contains a galaxy spanning every vertex except possibly the root.

Strong digraphs

Theorem : Spanning Galaxy Problem is polynomial-time solvable when restricted to strong digraphs.

The following problem is NP-complete:
Instance: A digraph D and an arc $a \in A(D)$.
Question: Does D have a spanning galaxy containing a?



The main theorem

Theorem

Given a strong digraph D, the following are equivalent:

- *D has a spanning galaxy.*
- *D contains an even strong subdigraph.*

even digraph: even number of vertices.

Even strong subdigraph– even cycle

It is poly-time solvable to decide if D as an even strong subdigraph.

Robertson, Seymour and Thomas — McCuaig :
It is poly-time solvable to decide if D as an even cycle.

Thomassen :
It is NP-complete to decide if D as an even cycle containing a prescribed arc.

The main theorem

Theorem

Given a strong digraph D, the following are equivalent:

(1) *D has a spanning galaxy.*

(2) *D contains a spanning winning arborescence.*

(3) *D has an even handle decomposition.*

(4) *D contains an even circuit or an even theta.*

(5) *D contains an even strong subdigraph.*

even digraph: even number of vertices.

Handle

handle in D: directed path $(s, v_1, \ldots, v_\ell, t)$ (with possibly $s = t$) such that:

- $d^-(v_i) = d^+(v_i) = 1$, for $1 \leq i \leq \ell$, and
- $D - \{v_1, \ldots, v_\ell\}$ is strong.

even handle: even number of arcs.

Handle decomposition

handle decomposition of D at v: $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$, with
$(D_i)_{0 \leq i \leq p}$ a sequence of strong digraphs and
$(h_i)_{1 \leq i \leq p}$ a sequence of handles such that:

- $V(D_0) = \{v\}$,
- h_i is a handle of D_i, for $1 \leq i \leq p$ and D_i is the edge disjoint union of D_{i-1} and h_i, and
- $D = D_p$.

even handle decomposition: one of its handles is even.

even strong subdigraph $\Rightarrow$ spanning galaxy (I)

even strong subdigraph $\Leftrightarrow$ even handle decomposition

$\Rightarrow$ D' smallest even strong subdigraph. Then any handle of D' is even (i.e. odd number of inner vertices).
Any handle decomposition of D' is even and can extended into an even handle decompositon of D.

$\Leftarrow$ As long as we add odd handles the parity stays the same.
At the first even handle the parity changes.

even strong subdigraph ⇒ spanning galaxy (II)

even handle decomposition ⇒ spanning galaxy

$(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$ even handle decomposition of D.
q: largest integer such that $h_q = (v_0, \ldots, v_{2k})$ is even.

D_{q-1} is strong so has a spanning arborescence T_{q-1} rooted at v_0.
Lemma ⇒ D_{q-1} has a galaxy G_{q-1} spanning all vertices but possibly v_0.

$G_q = G_{q-1} \cup \{v_0 v_1, v_2 v_3, \ldots v_{2k-2} v_{2k-1}\}$ spanning galaxy of D_q.

For $q < r \leq p$, $h_r = (x_0, x_1, \ldots, x_{2j}, x_{2j+1})$
$G_r = G_{r-1} \cup \{x_1 x_2, x_3 x_4, \ldots x_{2k-2} x_{2k-1}\}$ spanning galaxy of D_r.

Even strong digraph algorithm

$(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$ handle decomposition.
$h_q = (x_0, x_1, \ldots, x_\ell)$ last non-trivial handle.

Preprocessing $\Rightarrow$ no arc

- no arc from $V(D_{q-1})$ to $\{x_2, \ldots, x_{\ell-1}\}$;
- no arc $\{x_1, \ldots, x_{\ell-2}\}$ to $V(D_{q-1})$;
- no arc $x_i x_j$ with $j > i + 1$.

If the decomposition is even then $ESS(D) :=$"YES".
Else $D' := D_{q-1} +$ all arcs from $N^-_D(x_1)$ to $N^+_D(x_{\ell-1})$.
$ESS(D) := ESS(D')$ or $ESS(D[\{x_1, \ldots, x_{\ell-1}\}]$.

$ESS(D')$ computed recursively.
$ESS(D[\{x_1, \ldots, x_{\ell-1}\}]$ computed by a dedicated procedure.

Parameterized problems

parameterized problem: (problem, parameter)

(P, k) is FPT: admits an algorithm in time $O(f(k)n^c)$

k-Stars Spanning Galaxy Problem:
Instance: A digraph D and an integer k.
Parameter: k.
Question: Does D have a spanning galaxy with at most k stars?

Theorem : *k*-Stars Spanning Galaxy Problem is W[2]-complete.

Reduction from *k*-Domination.

Parameterized problems

k-Galaxy Problem:
Instance: A digraph D.
Parameter: k.
Question: Does D have a galaxy spanning at least k vertices?

Theorem : k-Galaxy Problem is FPT.

In fact: k-Galaxy Problem has a $(2k-2)$-kernelization.

Kernelization

$f(k)$-kernelization: polynomial-time algorithm
instance $(D, k) \longrightarrow$ instance (D', k') such that:

- (D', k') is equivalent to (D, k);
- $k' \leq k$ and $|D'| \leq f(k)$.

Kernelization + brute force = $O(g(k) + n^c)$ time algo.

locally maximal galaxy

$\text{lm}(G)$

(a) If $uv \in D \setminus G$, add uv to G.

(b) If $uv \in A(G)$, $uw \in A(D)$ and $w \notin V(G)$, add uw to G.

(c) If $tu \in E(G)$, $uv, uw \in A(D)$ and $v, w \in V(D) \setminus V(G)$, remove tu from G and add uv and uw.

(d) If $uv, uw \in E(G)$, $wx \notin A(D)$ and $x \notin V(G)$, remove uw from G and add wx.

Kernelization for k-Galaxy Problem

Ker(D, k)

Step 1: $G := (\emptyset; \emptyset)$;

Step 2: $G := \mathrm{Im}(G)$;

Step 3: If $|V(G)| \geq k$, return (S_k, k) .

Step 4: $N^+_G = \{v \in V(D) \setminus V(G) \mid \exists u \in V(G), uv \in A(D)\}$;
$N^-_G = V(D) \setminus (V(G) \cup N^+_G)$;

Step 5: Compute a maximum matching M in the bipartite graph induced by the arcs from $V(G)$ to $N^-(G)$;

Step 6: If $|V(M)| > |V(G)|$ then $G := M$ and go to Step 2;

Step 7: Return $(D\langle V(G) \cup N^+(G) \cup V(M)\rangle, k)$.

Directed star arboricity

directed star arboricity:
$dst(D)$ = minimum number of galaxies to cover $A(D)$.

Vizing: $dst(D) \leq \Delta(D) + 2$.

Conjecture 1: [Amini et al. 2007]
If $\Delta(D) \geq 3$ then $dst(D) \leq \Delta(D)$.

Tight if $\Delta^-(D) = \Delta$.

Amini et al.: Conjecture 1 holds when $\Delta = 3$.

Nice galaxy

nice galaxy: galaxy spanning all the vertices of maximum degree.

Conjecture 2:[Amini et al. 2007] $\Delta(D) \geq 4 \Rightarrow D$ has a nice galaxy.

Conjecture 2 (and so Conjecture 1) holds for acyclic digraphs.

Lemma : Every digraph has a galaxy spanning the vertices with indegree at least 2.

Theorem : $dst(D) \leq \Delta(D) + 1$.