Covering line graphs with equivalence relations

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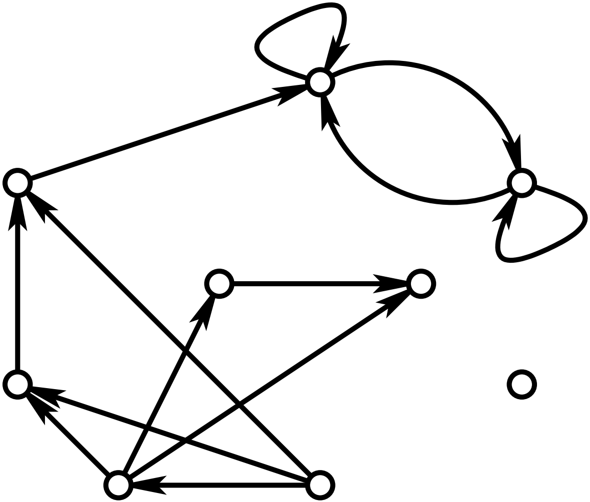
September 16, 2009

Original problem



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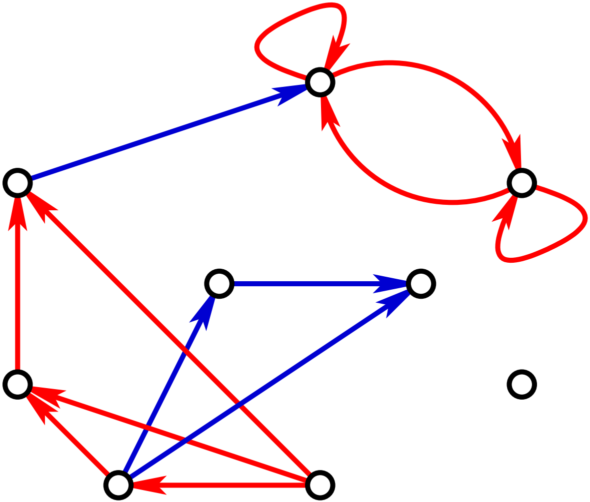
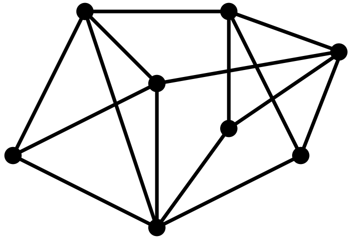


Equivalence covering



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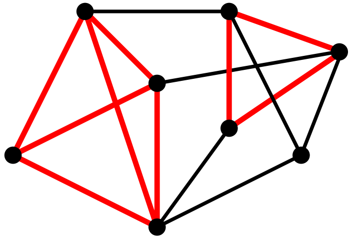

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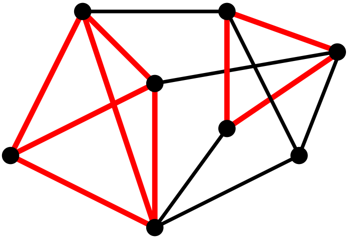

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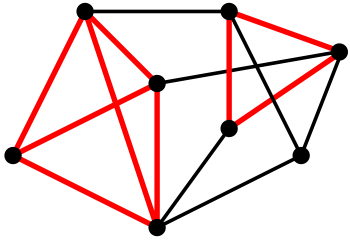

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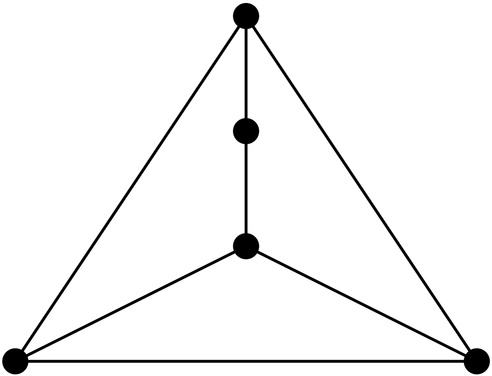

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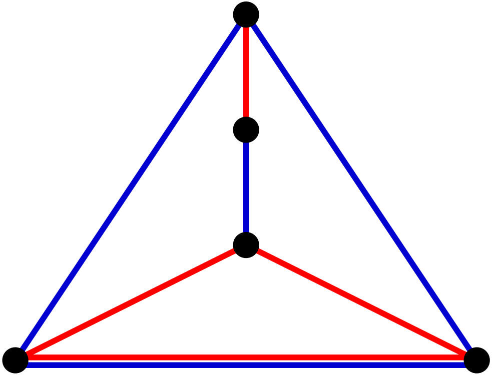

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Theorem (Alon 1986)

Let G be a graph on n vertices with minimum degree δ. Then

$$\log_2 n - \log_2(n - \delta - 1) \leq eq(G) \leq 2e^2(n-\delta)^2 \log_e n.$$

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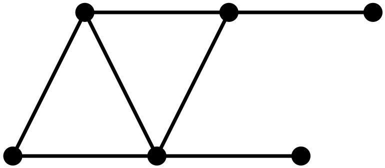

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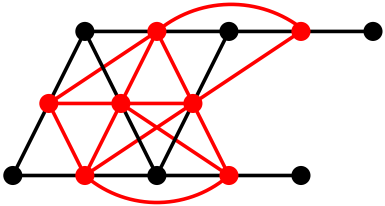

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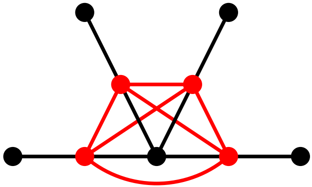

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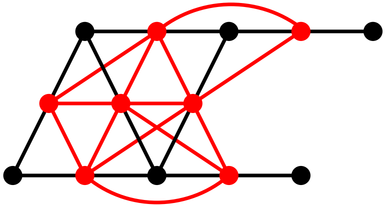

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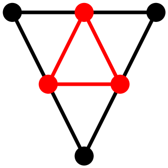

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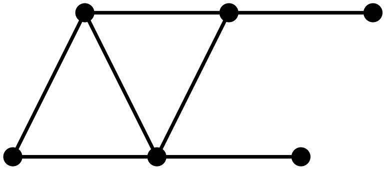
Theorem (E., Gimbel & King 2009)

For any graph G, $\frac{1}{3}\left(\log \log \chi(G) + 1\right) \leq eq(L(G)) \leq 2\left(\log \log \chi(G) + 1\right)$.

Orientations



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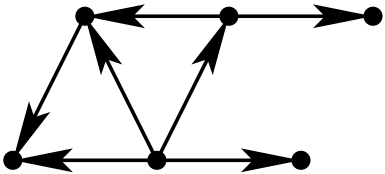

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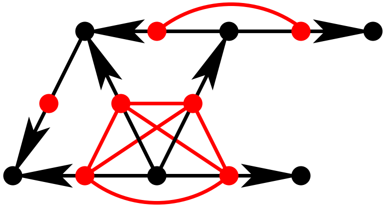

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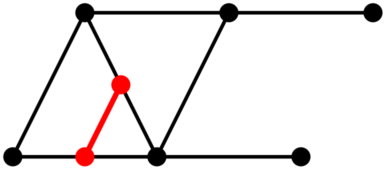

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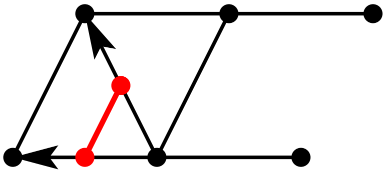

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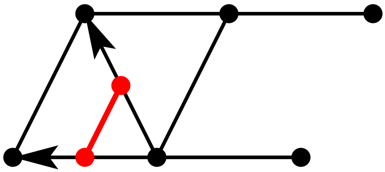

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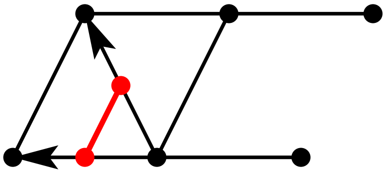

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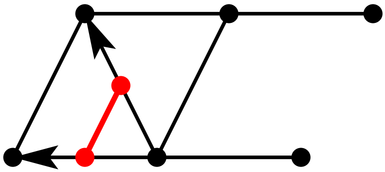

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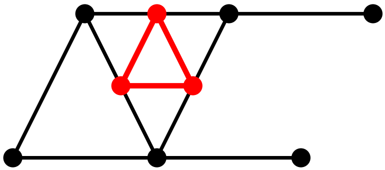

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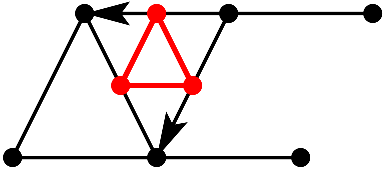

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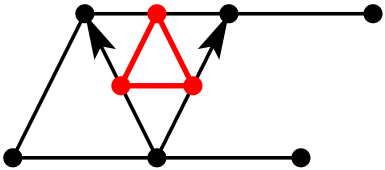

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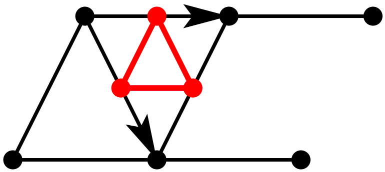

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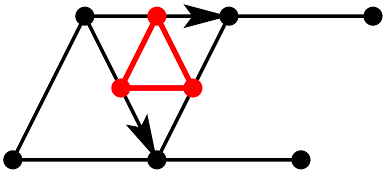

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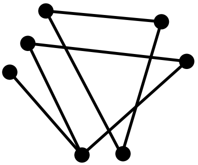

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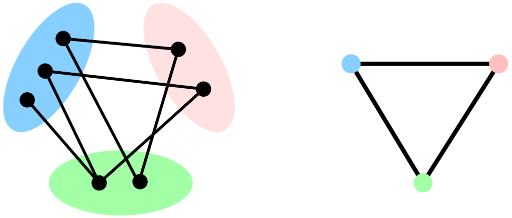

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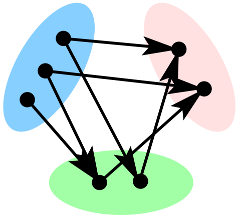

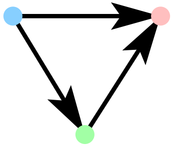

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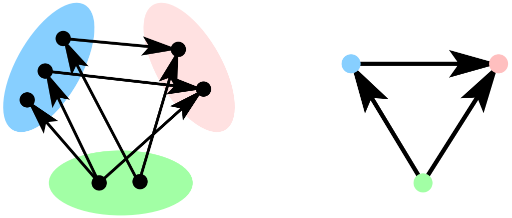

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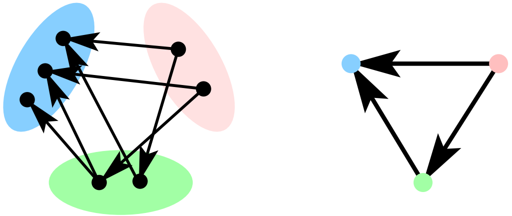

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Let G and H be graphs such that there is a homomorphism from G to H. Then $\sigma(G) \leq \sigma(H)$.

Consequence

For any graph G with chromatic number k, $\sigma(G) \leq \sigma(K_k)$.

The Upper Bound

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For any integer $n \geq 2$, $\sigma(K_n) \leq 2\left(\log_2 \log_2 n + 1\right)$.

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Any sequence of $2^{2^k} + 1$ vectors of $\mathbb{R}^k$ contains a monotonic subsequence of size 3 (here monotonic means monotonic in each coordinate).

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Theorem (De Bruijn 1940's)

There exists a sequence of 2^{2^k} vectors of $\mathbb{R}^k$ without monotonic subsequence of size 3.

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$$n = 2^{2^k}$$

k

11	**12**	**9**	**10**	**15**	**16**	**13**	**14**	**3**	**4**	**1**	**2**	**7**	**8**	**5**	**6**
4	**3**	**2**	**1**	**8**	**7**	**6**	**5**	**12**	**11**	**10**	**9**	**16**	**15**	**14**	**13**

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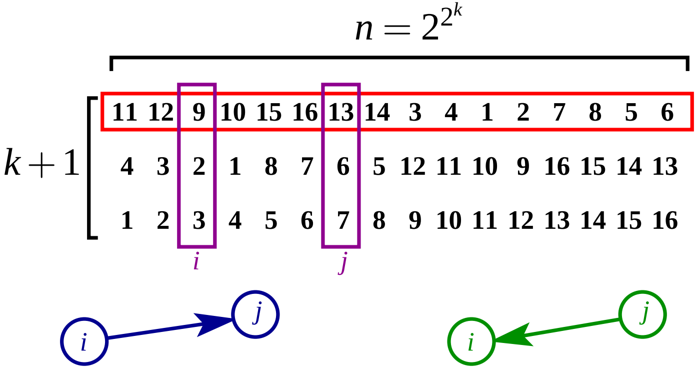

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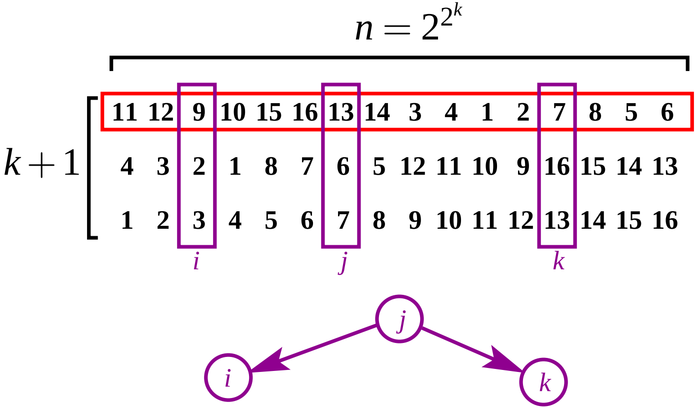

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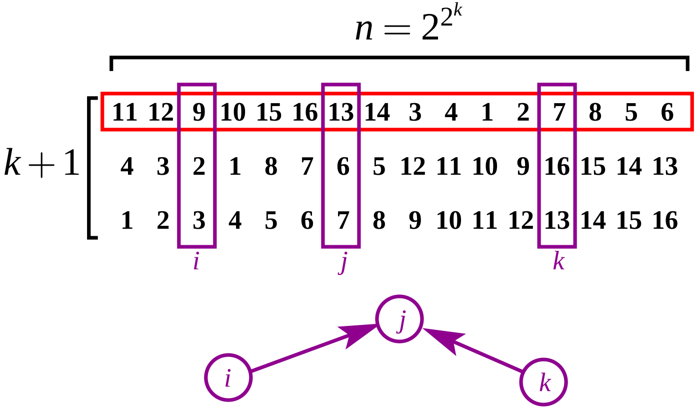

The Upper Bound

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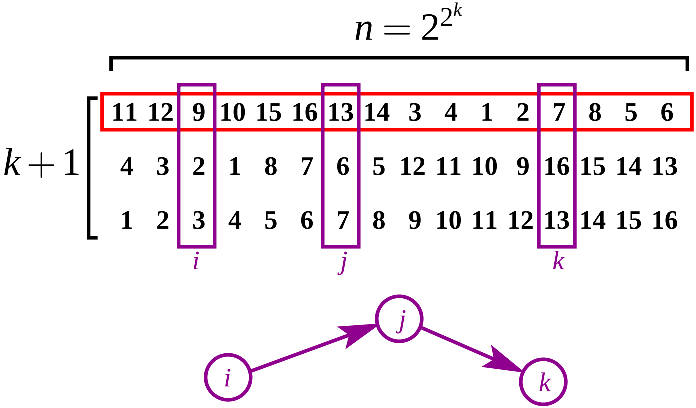

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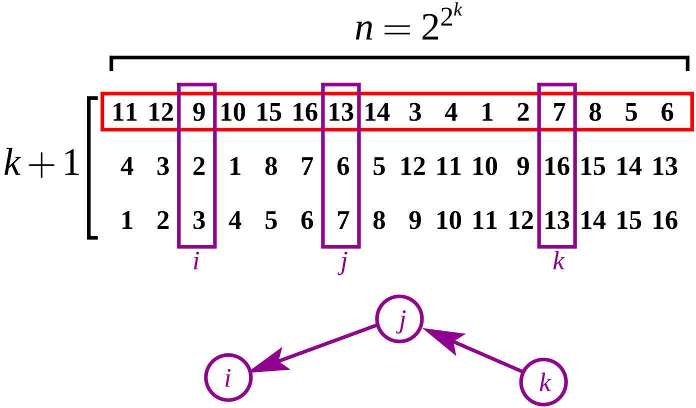

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For any graph G, $\chi(G) \leq 2^{2^{\sigma(G)-1}}$.

$1, \ldots, k$

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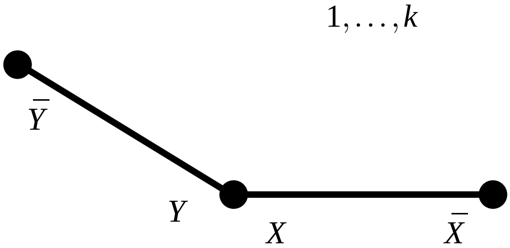

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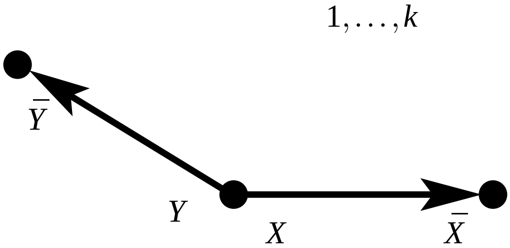

Conclusion

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For any graph G, $\log\log\chi(G) + 1 \leq \sigma(G) \leq 2\left(\log\log\chi(G) + 1\right).$

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Lemma

A graph G has $\sigma(G) = 3$ precisely if $3 \leq \chi(G) \leq 4$, and $\sigma(G) = 4$ precisely if $5 \leq \chi(G) \leq 12$.

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A graph G has $\sigma(G)=3$ precisely if $3 \le \chi(G) \le 4$, and $\sigma(G)=4$ precisely if $5 \le \chi(G) \le 12$.

Theorem

It is NP-complete to decide whether or not a triangle-free graph G has $\sigma(G) \le 3$ (resp. $\sigma(G) \le 4$). Equivalently, it is NP-complete to decide whether or not $eq(L(G)) \le 3$ (resp. $eq(L(G)) \le 4$).