Euler Complexes

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Abstract

We present a class of instances of the existence of a second object of a specified type, in fact, of an even number of objects of a specified type, which generalizes the existence of an equilibrium for bimatrix games. The proof is an abstract generalization of the Lemke-Howson algorithm for finding an equilibrium of a bimatrix game.

Keywords exchange algorithm, Euler complex, simplicial pseudo-manifold, room family, room partition, Euler graph, binary matroid, Euler binary matroid, Nash equilibrium, Lemke-Howson algorithm, matching algorithm, matroid partition algorithm.

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A *d-oik*, $C = (V, F)$, short for *d-dimensional Euler complex*, $d \geq 1$, is a finite set V of elements called the vertices of C and a family of $d+1$ element subsets of V, called the *rooms* of C, such that every d element subset of V is in an even number of the rooms.

A *wall of a room* means a set obtained by deleting one vertex of the room - and so any wall of a room in an oik is the wall of a positive even number of rooms of the oik.

Example 1. A d-dimensional simplicial pseudo-manifold is a d-oik where every d-element subset of vertices is in exactly zero or two rooms, i.e., in a simplicial pseudo-manifold any wall is the wall of exactly two rooms. An important special case of simplicial pseudo-manifold is a triangulation of a compact manifold such as a sphere.

Example 2. Let $Ax = b, x \geq 0$, be a tableau as in the simplex method, whose solution-set is bounded and whose basic feasible solutions are all non-zero (non-degenerate). Let V be the column-set of A. Let the rooms be the subsets S of columns such that $V - S$ is a feasible basis of the tableau. This is an $(n-r-1)$-oik where n is the number of columns of A and r is the rank (the number of rows) of A. In fact it is a triangulation of an $(n-r-1)$-dimensional sphere – in particular it is combinatorially the boundary of a 'simplicial polytope'.

Example 3. Let the n members of set V be colored with r colors. Let the rooms be the subsets S of V such that $V - S$ contains exactly one vertex of each color. This is an $(n-r-1)$-oik. In fact it is the oik of Example 2 where each column of A is all zeroes except for one positive entry.

Example 4. An Euler graph, that is a graph such that each of its vertices is in an even number of its edges (the rooms), is a 1-oik.

Example 5. For any connected Euler graph G with n vertices ($n \geq 3$), we have an $(n-2)$-oik (V, K) where V is the set of edges of G and the rooms are the edge-sets of the spanning trees of G.

Example 6. For any connected bipartite graph G with m edges and n vertices we have an $(m-n)$-oik where V is the edge-set of G, and the rooms are the edge-complements of spanning trees of G.

Example 7, generalizing Examples 5 and 6. Where M is an Euler binary matroid, that is a binary matroid of rank r such that each cocircuit, in fact each cocycle, is even, we have an $(r-1)$-oik, where V is the set of elements of the matroid, and the rooms are the bases of the matroid.

(A *binary matroid* M is given by a 0-1 matrix, A, mod 2. The elements of M are the columns. The bases of M are the linearly independent sets of columns. The cocycles are the supports of the row vectors generated by the rows of A. The cocircuits are the minimal cocycles. Matroid M is *Euler* when each row of A has an even number of ones. See, e.g., [4, 7].)

Let $M = [(V, F_i) : i = 1, , h]$ be an indexed collection of oiks (which we call an *oik-family*) all on the same vertex-set V.

The oiks of M are not necessarily of the same dimension. Of course, all of them may be the same oik.

A *room-family*, $R = [R_i : i = 1, , h]$, for oik-family M, is where, for each i , R_i is a room of oik i (i.e., a member of F_i). A *room-partition* R for M means a room-family whose rooms partition V, i.e., each vertex is in exactly one room of R.

Theorem 1 *Given an oik-family M and a room-partition R for M, there exists another different room-partition for M. In fact, for any oik-family M, there is an even number of room-partitions.*

Proof. Choose a vertex, say w, to be special. A *w-skew room-family for oik-family M* means a room-family, $R = [R_i : i = 1, , h]$, for M such that w is not in any of the rooms R_i, some vertex v is in exactly two of the R_i , and every other vertex is in exactly one of the R_i .

Consider the so-called exchange-graph X, determined by M and w, where the nodes of X are all the room-partitions for M and all the w-skew room-families for M. Two nodes of X are joined by an edge of X if each is obtained from the other by replacing one room by another. It is easy to see that the odd-degree nodes of X are all the room-partitions for M, and all the even-degree nodes of X are the w-skew room-families for M. Hence there is an even number of room-partitions for M. □

'Exchange algorithm': An algorithm for getting from one room-partition for M to another is to walk along a path in X, not repeating any edge of X, from one to another. Where each oik of the oik-family M is a simplicial pseudo-manifold, X consists of disjoint simple paths and simple cycles, and so the algorithm is uniquely determined by M and w.

Where oik-family M consists of two oiks of the kind in Example 2, the exchange algorithm is the Lemke-Howson algorithm for finding a Nash equilibrium of a 2-person game. Salvani and Von Stengel [6] show that the number of steps in the Lemke-Howson algorithm can grow exponentially relative to the size of the two tableaus of the game.

It is not known whether there is a polytime algorithm for finding a Nash equilibrium of a 2-person game. Chen and Deng [3] (see also [5]) proved a deep completeness result which is regarded as some evidence that there might not be a polytime algorithm.

Suppose each oik of M is given by an explicit list of its rooms, each oik perhaps a simplicial pseudo-manifold, perhaps a 2-dimensional sphere. Is some path of the exchange graph not well-bounded by the number of rooms?

How about the exchange algorithm when each oik of M is a 1-oik? If each oik of M is the same 1-oik then the well-known, non-trivial, non-bipartite matching algorithm [4, 7] can be used to find, if there is one, a first and a second room-partition.

How about the exchange algorithm where each oik of M is an Euler binary matroid? For an oik-family like that, the well-known, non-trivial, 'matroid partition' algorithm [4, 7] can be used to find, if there is one, a first and a second room-partition.

Example 8. A pure $(d+1)$-complex, $C = (V, F)$, means simply a finite set, V, and a family, F, of $d+2$ element subsets. The boundary, $bd(C) = (V, bd(F))$, of any pure $(d+1)$-complex, C, means the pure d-complex where $bd(F)$ is the family of those $d+1$ element subsets of V which are subsets of an odd number of members of F.

For any pure $(d+1)$-complex, C, its boundary, $bd(C)$, is a d-oik.

This is more-or-less the first theorem of simplicial homology theory. By recalling the meaning of d-oik, it is saying that for any pure $(d+1)$-complex, C, every d element subset, H, of V is a subset of an even number of $(d+1)$ element sets which are subsets of an odd number of the $(d+2)$ element members of F. It can be proved graph theoretically by observing that, for any d element subset, H, of V, the following graph, G, has an even number of odd degree vertices: The vertices of G are the $(d+1)$ element subsets of V which contain H. Two of these $(d+1)$ element vertices are joined by an edge in G when their union is a $(d+2)$ element member of F. Clearly a vertex of G is a subset of an odd number members of F, and hence is a member of $bd(F)$, when it is an odd degree vertex of G.

What can we say about $bd(F)$, besides Theorem 1, when F is the set of bases of a matroid?

In [2], different exchange graphs were studied. In [1], it was shown that Thomason's [8] exchange graph algorithm for finding a second hamiltonian circuit in a cubic graph is exponential relative to the size of the given graph.

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