Boxicity of graphs with bounded degree

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Abstract

The boxicity of a graph $G = (V, E)$ is the smallest $k$ for which there exist $k$ interval graphs $G_i = (V, E_i)$, $1 \leq i \leq k$, such that $E = E_1 \cap \ldots \cap E_k$. Graphs with boxicity at most $d$ are exactly the intersection graphs of (axis-parallel) boxes in $\mathbb{R}^d$. In this note, we prove that graphs with maximum degree $\Delta$ have boxicity at most $\Delta^2 + 2$, which improves the previous bound of $2\Delta^2$ obtained by Chandran et al (J. Combin. Theory Ser. B 98 (2008) 443–445).

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For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of subsets of a set $\Omega$, the intersection graph of $\mathcal{F}$ is defined as the graph with vertex set $\mathcal{F}$, in which two sets are adjacent if and only if their intersection is non-empty. A $d$-box is the cartesian product $[x_1, y_1] \times \ldots \times [x_d, y_d]$ of $d$ closed intervals of the real line. For any graph $G$, the boxicity of $G$, denoted by $\text{box}(G)$, is the smallest $d$ such that $G$ is the intersection graph of a family of $d$-boxes.

For a family of graphs $\{G_i = (V, E_i), 1 \leq i \leq k\}$ defined on the same vertex set, we set $G_1 \cap \ldots \cap G_k$ to be the graph with vertex set $V$, and edge set $E_1 \cap \ldots \cap E_k$, and we naturally say that the graph $G_1 \cap \ldots \cap G_k$ is the intersection of the graphs $G_1, \ldots, G_k$. The boxicity of a graph $G$ can be equivalently defined as the smallest $k$ such that $G$ is the intersection of $k$ interval graphs. Graphs with boxicity one are exactly interval graphs, which can be recognized in linear time. On the other hand, Kratochvíl [5] proved that determining whether $\text{box}(G) \leq 2$ is NP-complete.

The concept of boxicity was introduced in 1969 by Roberts [8]. It is used as a measure of the complexity of ecological [9] and social [4] networks, and has applications in fleet maintenance [7]. Boxicity has been investigated for various classes of graphs.

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[3, 10, 11], and has been related with other parameters, such as treewidth [1]. Recently, Chandran et al. [2] proved that every graph with maximum degree at most \( \Delta \) has boxicity at most \( 2\Delta^2 \). To prove this bound, Chandran et al. use the fact that if a graph \( G \) is the intersection of \( k \) graphs \( G_1, \ldots, G_k \), we have \( \text{box}(G) \leq \sum_{1 \leq i \leq k} \text{box}(G_i) \).

In the remaining of the article, we use the same idea to prove the following theorem:

**Theorem 1** Every graph with maximum degree \( \Delta \) has boxicity at most \( 2 \left\lfloor \frac{\Delta^2}{2} \right\rfloor + 2 \).

**Proof.** Let \( G = (V, E) \) be a graph with maximum degree \( \Delta \), and let \( c \) be a (not necessarily proper) coloring of the vertices of \( G \) with colors from \( \{1, \ldots, 2k\} \) such that:

(i) there is no path \( uvw \) with \( c(u) = c(w) \);

(ii) for any \( 1 \leq j \leq k \), there is no edge between a vertex colored with \( 2j - 1 \) and a vertex colored with \( 2j \).

Observe that condition (i) implies that the graph induced by each color class is a graph with maximum degree at most one (the disjoint union of a stable set and a matching). The first step of the proof is to find the smallest \( k \) such that a \( 2k \)-coloring as defined above exists. Define the function \( f \) such that for every \( j \geq 1 \), \( f(2j) = 2j - 1 \) and \( f(2j - 1) = 2j \). We color the vertices of \( G \) one by one with the following procedure: while coloring a vertex \( u \in V \), we choose for \( u \) a color from \( \{1, \ldots, 2k\} \setminus (N_1 \cup N_2) \), where \( N_1 = \{f(c(v)) \mid v \text{ is a colored neighbor of } u\} \) and \( N_2 = \{c(v) \mid u \text{ and } v \text{ have a common (not necessarily colored) neighbor}\} \).

If we follow this procedure, the partial coloring obtained at the end of each step has the desired properties: since \( c(u) \notin N_1 \), condition (ii) is still verified, and since \( c(u) \notin N_2 \), condition (i) is also still verified. At each step, \( N_1 \) has size at most \( \Delta \) and \( N_2 \) has size at most \( \Delta(\Delta - 1) \). Hence if \( k = \left\lceil \frac{\Delta^2 + 1}{2} \right\rceil = \left\lceil \frac{\Delta^2}{2} \right\rceil + 1 \), a \( 2k \)-coloring of \( G \) as defined above exists.

From now on, we assume that \( k = \left\lfloor \frac{\Delta^2}{2} \right\rfloor + 1 \). Hence, a \( 2k \)-coloring \( c \) of \( G \) with the properties defined above exists. For any \( 1 \leq i \leq k \), let \( G_i \) be the graph obtained from \( G \) by adding an edge between any two non-adjacent vertices \( u, v \) such that \( c(u), c(v) \notin \{2i - 1, 2i\} \). Using conditions (i) and (ii), \( G_i \) can be decomposed into a clique \( K_i \) (induced by the vertices colored neither with \( 2i - 1 \), nor with \( 2i \)), and two sets \( S_{2i-1} \) and \( S_{2i} \) corresponding to the vertices colored with \( 2i - 1 \) and \( 2i \) respectively (see Figure 2(a)). By condition (ii), there is no edge between \( S_{2i-1} \) and \( S_{2i} \), and by condition (i), every vertex of \( K_i \) is adjacent to at least one vertex of \( S_{2i-1} \) and one vertex of \( S_{2i} \). Moreover, \( S_{2i-1} \) and \( S_{2i} \) both induce a graph with maximum degree one by condition (i).
Now observe that \( G = \cap_{1 \leq i \leq k} G_i \). If two vertices are adjacent in \( G \) they are also adjacent in any \( G_i \), since \( G \subseteq G_i \). On the other hand, if two vertices \( u \) and \( v \) are not adjacent in \( G \), then they are not adjacent in \( G_{[c(u)/2]} \), and so they are not adjacent in the intersection of the \( G_i \)-s.

As a consequence, \( \text{box}(G) \leq \sum_{1 \leq i \leq k} \text{box}(G_i) \). We now show that every graph \( G_i \) has boxity at most two, which implies that \( \text{box}(G) \leq 2([\Delta^2/2] + 1) \) and concludes the proof.

![Figure 1: The ordering of the vertices of \( S_{2i-1} \) and \( S_{2i} \).](image)

For any \( 1 \leq i \leq k \), we represent \( G_i \) as the intersection graph of 2-dimensional boxes. We order the vertices \( u_1, \ldots, u_s \) of \( S_{2i-1} \) and the vertices \( v_1, \ldots, v_t \) of \( S_{2i} \) as depicted in Figure 1 (recall that \( S_{2i-1} \) and \( S_{2i} \) both induce a graph with maximum degree at most one). Let \( r \) be the maximum of \( s \) and \( t \). For every \( j \) such that \( u_{2j-1} \) and \( u_{2j} \) are adjacent in \( S_{2i-1} \), \( u_{2j-1} \) is represented by the box \( [-r+2j-1] \times [-2j+2,-2j+1] \) and \( u_{2j} \) is represented by the box \( [-r+2j-1,-r+2j] \times [-2j+1] \). If a vertex \( u_j \) is isolated in \( S_{2i-1} \), it is represented by the point \((-r+j,-j+1)\).

Similarly, for every \( j \) such that \( v_{2j-1} \) and \( v_{2j} \) are adjacent in \( S_{2i} \), \( v_{2j-1} \) is represented by the box \([2j-2,2j-1] \times \{r-2j+1\} \) and \( v_{2j} \) is represented by the box \([2j-1] \times \{r-2j,r-2j+1\} \). If a vertex \( v_j \) is isolated in \( S_{2i} \), it is represented by the point \((j-1,r-j)\) (see Figure 2(b) for an example).

Observe that:

1. the boxes of two adjacent vertices \( u_{2j-1} \) and \( u_{2j} \) intersect in \((-r+2j-1,-2j+1)\);
2. the boxes of two adjacent vertices \( v_{2j-1} \) and \( v_{2j} \) intersect in \((2j-1,r-2j+1)\);
3. the boxes of all the other pairs of vertices colored with \( 2i-1 \) or \( 2i \) are not intersecting.
4. the top-right corner of the box of \( u_j \) is the point \((-r+j,-j+1)\) and the bottom-left corner of the box of \( v_j \) is the point \((j-1,r-j)\)

We now have to represent the vertices from \( K_i \). We represent the vertices having no neighbor outside \( K_i \) by the point \((0,0)\). If a vertex \( u \) from \( K_i \) has only one neighbor outside \( K_i \), say \( u_j \in S_{2i-1} \), we represent \( u \) by the box \([-r+j,0] \times [-j+1,0] \). If a vertex \( v \) from \( K_i \) has only one neighbor outside \( K_i \), say \( v_j \in S_{2i} \), we represent \( v \) by the box \([0,j-1] \times [0,r-j] \). If a vertex \( w \) of \( K_i \) has one neighbor \( u_j \in S_{2i-1} \) and
one neighbor \( v_ℓ ∈ S_{2i} \), we represent \( w \) by the box \([-r+j, ℓ-1] × [-j+1, r-ℓ]\) (see Figure 2(b) for an example).

The boxes representing the vertices from \( K_i \) are pairwise intersecting, since they all contain the point \((0, 0)\). Moreover, using Observation (4) above, the box of every vertex \( v \) from \( K_i \) only intersects the boxes of the neighbors of \( v \). Hence, \( G_i \) is the intersection graph corresponding to this representation, and so \( G_i \) has boxicity two, which concludes the proof.

The best known lower bound for the boxicity of graphs with maximum degree \( Δ \) was given by Roberts [8]. Consider the graph \( H_{2n} \) obtained by removing a perfect matching from a clique of \( 2n \) vertices. If this graph has boxicity \( k ≤ n - 1 \), let \( G_1, \ldots, G_k \) be interval graphs such that \( H_{2n} = G_1 ∩ \ldots ∩ G_k \). Since \( k ≤ n - 1 \) and \( H_{2n} \) have \( n \) non-edges, two non-edges of \( H_{2n} \) have to lie in the same interval graph, say \( G_i \). This is impossible since otherwise \( G_i \) contains an induced cycle of length four and is not an interval graph. Hence, \( \text{box}(H_{2n}) ≥ n ≥ \lceil \frac{1}{2} Δ(H_{2n}) \rceil \).

Chandran et al. [2] conjectured that for any graph \( G \), \( \text{box}(G) ≤ O(Δ) \). It is interesting to remark that this conjecture is true when the graphs \( G_1, \ldots, G_k \) with \( G = ∩_{1 ≤ i ≤ k} G_i \) are only required to be chordal. McKee and Scheinerman [6] defined the *chordality* of a graph \( G \), denoted by \( \text{chord}(G) \), as the smallest \( k \) such that \( G \) is the intersection of \( k \) chordal graphs. Since a graph is an interval graph if and

![Figure 2: (a) A graph \( G_i \) and (b) a representation of \( G_i \) as the intersection graph of 2-dimensional boxes.](image-url)
only if it is chordal and its complement is a comparability graph, we clearly have chord\((G) \leq \text{box}(G)\) for any graph \(G\). McKee and Scheinerman proved that the chordality of a graph is bounded by its chromatic number. As a corollary, it is easy to show that for any graph \(G\) with maximum degree \(\Delta\), chord\((G) \leq \Delta\).

We conclude with general remarks. We denote by \(a(G)\) the arboricity of \(G\), that is the minimum number of induced forests into which the edges of \(G\) can be partitioned. For outerplanar graphs, planar graphs, graphs with bounded treewidth, and graphs with bounded degree, the boxicity seems to be bounded by the arboricity. Unfortunately it seems to be false in general: there exists trees with boxicity at least two, and graphs with arboricity two and boxicity at least three. This leads to two natural questions:

1. is there a constant \(\kappa \geq 1\), such that any graph \(G\) satisfies box\((G) \leq a(G) + \kappa\)?
2. is there a constant \(\lambda > 1\), such that any graph \(G\) satisfies box\((G) \leq \lambda a(G)\)?

A positive answer to the second question (and thus to the first), would imply that for any graph \(G\) with maximum degree \(\Delta\), box\((G) \leq \lambda \left\lceil \frac{\Delta + 1}{2} \right\rceil\), proving the conjecture of Chandran et al. [2].

References


