The Structure and Chi-Boundedness of Typical Graphs in a Hereditary Family

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The Structure of My Talk

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Part I
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Talking about not talking about the known unknowns.
Bounding The Chromatic Number of Graphs in a Hereditary Family
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\[ \mathcal{F} \] is hereditary if \( G \in \mathcal{F} \) & \( H \subseteq G \Rightarrow H \in \mathcal{F} \]
Bounding The Chromatic Number of Graphs in a Hereditary Family

$\mathcal{F}$ is hereditary if $G \in \mathcal{F}$ & $H \subseteq G \Rightarrow H \in \mathcal{F}$

$\chi(G)$ is the minimum number of colours needed to colour the vertices of $G$ so that no edge is monochromatic.
χ-Boundedness

\( \mathcal{F} \) is χ-bounded if there is some function \( b_\mathcal{F} \) such that for every \( G \in \mathcal{F} \) we have:

\[ \chi(G) \leq b_\mathcal{F}(\omega(G)) \]

where \( \omega(G) \) is the size of the largest clique in \( G \).
Berge’s Perfection

Every graph in $\mathcal{F}$ satisfies $\chi=\omega$ precisely if $\mathcal{F}$ contains no odd cycle of length at least five and no complement of such a cycle.

*Chudnovsky, Robertson, Seymour, Thomas 2006*
The $\chi$-Boundness of H-free Graphs
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If $H$ is not a forest then the H-Free graphs are not $\chi$-bounded. *Erdos 1959.*
The $\chi$-Boundness of H-free Graphs

If $H$ is not a forest then the $H$-Free graphs are not $\chi$-bounded. *Erdos 1959*

**Gyarfas’ Conjecture:** For every forest $T$, the $T$-free graphs are $\chi$-bounded. *Gyarfas 1987*
Forbidding Long Odd Cycles

Theorem: $\mathcal{F}_1$ obtained by excluding the complements of odd cycles of length at least 5 is not $\chi$-bounded.
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Conjecture: $\mathcal{F}_2$ obtained excluding the odd cycles of length at least 5 is $\chi$-bounded. Gyarfas 1987
Forbidding Long Odd Cycles

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Conjecture: $\mathcal{F}_2$ obtained excluding the odd cycles of length at least 5 is $\chi$-bounded. *Gyarfas 1987*

The Hoang-McDiarmid Conjecture: The vertices of every graph in $\mathcal{F}_2$ can be 2-coloured so that no maximum clique is monochromatic.
Conjecture: For all l, a hereditary family containing no odd cycle of length at least l, is \(\chi\)-bounded. Gyarfas 1987
Forbidding An Infinite Set of Cycles

Conjecture: For all \( l \), a hereditary family containing no odd cycle of length at least \( l \), is \( \chi \)-bounded. *Gyarfas 1987*

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Theorem: For all $l$, a hereditary family containing contains no odd cycle of length at least 5 and no cycle of length at least $l$ is $\chi$-bounded. *Scott 1999*
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Theorem: If $G$ contains no even induced cycle then $\chi(G) < 2\omega(G)$. *Addarrio-Berry, Chudnovsky, Havet, Reed, Seymour 2008.*
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Scott’s Example: There are hereditary families which exclude an infinite number of cycles and are not $\chi$-bounded
Part II: Typical Behaviour of $\chi$ in A Hereditary Family

Almost every graph with no induced cycle of length $>4$ satisfies $\chi = \omega$. Promel and Steger 1992.

For every tree $T$, almost every $T$-free $G$ satisfies $\chi = \omega$. Reed and Yuditsky 2014.

For every set of cycles not including $C_3$ or $C_6$, almost every graph containing none of these cycles as induced subgraphs satisfies $\chi = \omega$. Reed and ScoG 2014.

For every $H$, there is function $b_H$ such that most of the graphs which contain neither $H$ nor its complement as an induced subgraph satisfy $\chi \leq b_H(\omega)$. Furthermore, for almost every $H$, we can take $b_H(\omega) < 2\omega \log |V(H)|$. Loebl, Reed, ScoG, Thomase, Thomason 2010 & Kang, Reed, McDiarmid 2014.
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For every $H$, there is function $b_H$ such that most of the graphs which contain neither $H$ nor its complement as an induced subgraph satisfy $\chi \leq b_H(\omega)$. Furthermore, for almost every $H$, we can take $b_H(\omega)=\omega(\omega|V(H)|/\log |V(H)|)$.

*Loebl, Reed, Scott, Thomasse, Thomason 2010.*

& *Kang, Reed, McDiarmid 2014*
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For $k>3$, a.e. $C_{2k+1}$-free graph can be partitioned into $k$ cliques. 
*Balogh and Butterfield 2011*
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For $k>3$, a.e. $C_{2k+1}$-free graph can be partitioned into $k$ cliques.
Balogh and Butterfield 2011

For $k>5$ a.e. $C_{2k}$-free graph can be partitioned into $k-2$ cliques and a graph whose complement is the disjoint union of triangles and stars.
Reed and Scott 2014
A Conjecture

The witnessing parameter $\omega_p(H)$, is the maximum $t$ such that for some $a+b=t$, there is no partition of $H$ into $a$ cliques and $b$ stable sets.

A witnessing partition of $H$-freeness is a partition of the vertex set of $G$ into $S_1, \ldots, S_{\omega_p(H)}$ such that for any partition of $V(H)$ into $X_1, \ldots, X_{\omega_p(H)}$, there is an $i$ for which $H[X_i]$ is not an induced subgraph of $G[S_i]$.

Conjecture: For all $H$, almost every $H$-free graph has a witnessing partition of $H$-freeness.
A Conjecture

The *witnessing partition number of* $H$, $wpn(H)$, is the maximum $t$ such that for some $c+s=t$, there is no partition of $H$ into $c$ cliques and $s$ stable sets.
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The *witnessing partition number of H, wpn(H)*, is the maximum $t$ such that for some $c+s=t$, there is no partition of $H$ into $c$ cliques and $s$ stable sets.

A *witnessing partition of H-freeness* is a partition of the vertex set of $G$ into $S_1, \ldots, S_{wpn(H)}$ such that for any partition of $V(H)$ into $X_1, \ldots, X_{wpn(H)}$, there is an $i$ for which $H[X_i]$ is not an induced subgraph of $G[S_i]$. 
A Conjecture

The *witnessing partition number* of $H$, $\text{wpn}(H)$, is the maximum $t$ such that for some $c+s=t$, there is no partition of $H$ into $c$ cliques and $s$ stable sets.

A *witnessing partition of $H$-freeness* is a partition of the vertex set of $G$ into $S_1, \ldots, S_{\text{wpn}(H)}$ such that for any partition of $V(H)$ into $X_1, \ldots, X_{\text{wpn}(H)}$ there is an $i$ for which $H[X_i]$ is not an induced subgraph of $G[S_i]$.

Conjecture: For all $H$, almost every $H$-free graph has a witnessing partition of $H$-freeness.
Triangle Free Graphs

There are least $2^{n^2/4}$ bipartite graphs on \{1,\ldots,n\}. 
Coloured Regularity Partition Cliques
Coloured Regularity Partition Clique

For an equi-partition $X_1,...,X_p$ of $V(G)=\{1,...,n\}$, an edge $ij$ of the clique on $\{1,...,p\}$ is:

Grey if $(X_i,X_j)$ is irregular,
Blue if $(X_i,X_j)$ is regular with density near 0,
Green Otherwise

Every large enough $G$ has a partition with $o(p^2)$ grey edges for $p$ as large as we want but fixed. Szemerédi
Coloured Regularity Partition Cliques

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If $G$ is $\Delta$-free there is no triangle whose edges are green.
Coloured Regularity Partition Clique
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For an equi-partition $X_1, \ldots, X_p$ of $V(G) = \{1, \ldots, n\}$, an edge $ij$ of the clique on $\{1, \ldots, p\}$ is:

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- Blue if $(X_i, X_j)$ is regular with density near 0,
- Green Otherwise

Every large enough $G$ has a partition with $o(p^2)$ grey edges for $p$ as large as we want but fixed. Szemeredi

If $G$ is $\Delta$-free there is no triangle whose edges are green. Hence at most half the edges of the edges are green.
Triangle-Free Graphs:

A Rough Partition

The number of graphs permitting a partition with $\binom{p}{2}$ green edges is $2^{(1+o(1))}p_g\binom{n}{2}$
Triangle-Free Graphs: A Rough Partition

The number of graphs permitting a partition with $p_g\binom{p}{2}$ green edges is $2^{(1+o(1))p_g\binom{n}{2}}$

Almost every $H$-free graph permits a partition with $p_g=\frac{1+o(1)}{2}$
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Almost every H-free graph permits a partition with $p_g=\frac{1+o(1)}{2}$

The associated green graph is $\Delta$-free, so contains a bipartite graph with $\frac{(1+o(1))\binom{p}{2}}{2}$ edges.
Triangle-Free Graphs: A Rough Partition

The number of graphs permitting a partition with \( p_g\binom{p}{2} \) green edges is \( 2^{(1+o(1))p_g\binom{n}{2}} \).

Almost every \( H \)-free graph permits a partition with \( p_g = \frac{1+o(1)}{2} \).

The associated green graph is \( \Delta \)-free, so contains a bipartite graph with \( \frac{(1+o(1))}{2} \binom{p}{2} \) edges.

This provides a bipartition of \( \{1,\ldots,n\} \) with \( o(n^2) \) edges of \( G \) within each side.
Triangle-Free Graphs: Refining The Partition

For each partition of \{1,\ldots,n\} into A and B there are $2^{o(n^2)}$ choices for subgraphs $G[A]$ and $G[B]$ with $o(n^2)$ edges. For each such choice there are at most $2^{|A||B|}$ choices for $E(A,B)$. 
Triangle-Free Graphs: Refining The Partition

For each partition of \{1,\ldots,n\} into A and B there are \(2^{o(n^2)}\) choices for subgraphs \(G[A]\) and \(G[B]\) with \(o(n^2)\) edges. For each such choice there are at most \(2^{|A||B|}\) choices for \(E(A,B)\).

So, almost every \(\Delta\)-free graph permits a partition such that:

(i) \(|A| - n/2 = o(n)\).
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For each partition of \{1,...,n\} into A and B there are $2^{o(n^2)}$ choices for subgraphs $G[A]$ and $G[B]$ with $o(n^2)$ edges. For each such choice there are at most $2^{|A||B|}$ choices for $E(A,B)$.

So, almost every $\Delta$-free graph permits a partition such that:

(i) $|A|-n/2=o(n)$.
(ii) $A$ contains a stable set of size $|A|(1-o(1))$,
(iii) $B$ contains a stable set of size $|B|(1-o(1))$,.
Triangle-Free Graphs: Refining The Partition

For each partition of \( \{1,\ldots,n\} \) into A and B there are \( 2^{o(n^2)} \) choices for subgraphs \( G[A] \) and \( G[B] \) with \( o(n^2) \) edges. For each such choice there are at most \( 2^{|A||B|} \) choices for \( E(A,B) \).

So, almost every \( \Delta \)-free graph permits a partition such that:

(i) \( |A|-n/2=o(n) \).
(ii) A contains a stable set of size \( |A|(1-o(1)) \),
(iii) B contains a stable set of size \( |B|(1-o(1)) \),
(iv) no vertex has more than \( n/100 \) neighbours in both A and B
Triangle-Free Graphs: Refining The Partition

For each partition of \( \{1, \ldots, n\} \) into \( A \) and \( B \) there are \( 2^{o(n^2)} \) choices for subgraphs \( G[A] \) and \( G[B] \) with \( o(n^2) \) edges. For each such choice there are at most \( 2^{|A||B|} \) choices for \( E(A,B) \).

So, almost every \( \Delta \)-free graph permits a partition such that:

(i) \( |A|-n/2=o(n) \).

(ii) \( A \) contains a stable set of size \( |A|(1-o(1)) \),

(iii) \( B \) contains a stable set of size \( |B|(1-o(1)) \),

(iv) no vertex has more than \( n/100 \) neighbours in both \( A \) and \( B \).
Coloured Regularity Partition Cliques Revisited

For an equi-partition $X_1,\ldots,X_p$ of $V(G)=\{1,\ldots,n\}$, an edge $ij$ of the clique on $\{1,\ldots,p\}$ is:

Grey if $(X_i,X_j)$ is irregular,
Red if $(X_i,X_j)$ is regular with density near 1,
Blue if $(X_i,X_j)$ is regular with density near 0,
Green Otherwise

If $G$ is $H$-free there is no injection $f:V(H)\rightarrow\{1,\ldots,p\}$ such that for each edge $uv$ of $H$, $f(u)f(v)$ is green or red, and for each non-edge $uv$ $f(u)f(v)$ is green or blue.
Coloured Regularity Partition Clique Revisited

For an equi-partition $X_1,\ldots,X_p$ of $V(G)=\{1,\ldots,n\}$, an edge $ij$ of the clique on $\{1,\ldots,p\}$ is:

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If $G$ is H-free there is no injection $f:V(H)\rightarrow\{1,\ldots,p\}$ such that for each edge $uv$ of $H$, $f(u)f(v)$ is green or red, and for each non-edge $uv$ $f(u)f(v)$ is green or blue.
This implies that there is no green wpn$(H)+1$ clique in a coloured partition clique corresponding to $G$. 
H-free Graphs: A Rough Partition

For some $c+s=\up{wpmn}(H)$, there are $2^{\left(1-\frac{1}{\up{wpmn}(H)}+o(1)\right)\binom{n}{2}}$ H-free graphs on $\{1,\ldots,n\}$ whose vertices can be partitioned into $c$ cliques and $s$ stable sets.
H-free Graphs: A Rough Partition

For some $c+s=\text{wpn}(H)$, there are $2^{\left(1-\frac{1}{\text{wpn}(H)}+o(1)\right)\binom{n}{2}}$ H-free graphs on $\{1,\ldots,n\}$ whose vertices can be partitioned into $c$ cliques and $s$ stable sets.

The vertices of a.e. H-free graph on $\{1,\ldots,n\}$ can be equipartitioned into $\text{wpn}(H)$ sets such that each partition class permits a Szemeredi partition permitted by $2^{o(\binom{n}{2})}$ graphs.
A Better Rough Partition

(Reed,Scott cf. Alon,Balogh,Bollobas,Morris)
A Better Rough Partition

(Reed, Scott cf. Alon, Balogh, Bollobas, Morris)

For every $H, \delta$ there are $\varepsilon, b > 0$ s.t. for a.e. $H$-free graph, $V(G)$ has a partition into $S_1, ..., S_{\text{wpn}(H)}$ for which there are two exceptional sets $Z$ and $B$ with $|Z| \leq n^{2-\varepsilon}$ and $|B| \leq b$ such that:

(i) $(S_1 - Z, ..., S_{\text{wpn}(H)} - Z)$ is an $H$-witnessing partition of $G - Z$
A Better Rough Partition  
(Reed,Scott cf. Alon,Balogh,Bollobas,Morris)

For every $H, \delta$ there are $\varepsilon, b > 0$ s.t. for a.e. $H$-free graph, $V(G)$ has a partition into $S_1, \ldots, S_{\wpn(H)}$ for which there are two exceptional sets $Z$ and $B$ with $|Z| \leq n^{2-\varepsilon}$ and $|B| \leq b$ such that:

(i) $(S_1 - Z, \ldots, S_{\wpn(H)} - Z)$ is an $H$-witnessing partition of $G-Z$

(ii) for every $v$ in $S_i$ there is a $w$ in $B$ such that:

$$|S_i \cap ((N(v) - N(w)) \cup (N(w) - N(v)))| \leq \delta n$$
A Better Rough Partition
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For every $H, \delta$ there are $\epsilon, b > 0$ s.t. for a.e. $H$-free graph, $V(G)$ has a partition into $S_1, \ldots, S_{\text{wpn}(H)}$ for which there are two exceptional sets $Z$ and $B$ with $|Z| \leq n^{2-\epsilon}$ and $|B| \leq b$ such that:

(i) $(S_1 - Z, \ldots, S_{\text{wpn}(H)} - Z)$ is an $H$-witnessing partition of $G - Z$

(ii) for every $v$ in $S_i$ there is a $w$ in $B$ such that:

$$|S_i \cap ((N(v) - N(w)) \cup (N(w) - N(v)))| \leq \delta n$$

This implies that for every $S_i$, there are bipartite, split, and cobipartite subgraphs of $H$ which are not induced subgraphs of $G[S_i - Z]$. Hence the number of choices for $G[S_i]$ is at most $2$ to the $n^{2-\epsilon}$. 

Using (ii) for every $v$ in $S_i$ there is a $w$ in $B$ such that:

$$|S_i \cap ((N(v) - N(w)) \cup (N(w) - N(v)))| \leq \delta n$$
More on This Partition

(Kang, McDiarmid, Reed)

For every $\delta$, for almost every $H$ for all but (i) $(S_1 - Z, \ldots, S_{\text{wpn}(H)} - Z)$ is an $H$-witnessing partition of $G - Z$ in which $o(\text{wpn}(H))$ of the $S_i - Z$ are obtained from a split graph by substituting stable sets for the vertices of the clique and cliques for the vertices of the stable set,
Possible Partitions for $C_6$-Free Graphs

$C_6$ can be partitioned into two stable sets, a stable set and two cliques, and three cliques but not two cliques so $\text{wpn}(C_6)$ is 2.
Possible Partitions for $C_6$-Free Graphs

$C_6$ can be partitioned into two stable sets, a stable set and two cliques, and three cliques but not two cliques so $wpn(C_6)$ is 2.

If $S_i-Z$ is a clique then $S_{2-i}-Z$ must be $P_4$-free.
Possible Partitions for $C_6$-Free Graphs

$C_6$ can be partitioned into two stable sets, a stable set and two cliques, and three cliques but not two cliques so \( \text{w}p\text{n}(C_6) \) is 2.

If \( S_i-Z \) is a clique then \( S_{2-i}-Z \) must be $P_{4}$-free.

If \( S_i-Z \) is a stable set then \( S_{2-i}-Z \) must be the complement of a graph of girth 5.
Possible Partitions for $C_6$-Free Graphs

$C_6$ can be partitioned into two stable sets, a stable set and two cliques, and three cliques but not two cliques so $\text{wpn}(C_6)$ is 2.

If $S_i-Z$ is a clique then $S_{2-i}-Z$ must be $P_4$-free.

If $S_i-Z$ is a stable set then $S_{2-i}-Z$ must be the complement of a graph of girth 5.

Otherwise, since not both $S_{1}-Z$ and $S_{2}-Z$ contain a copy of $F$ if $F$ is $P_3$, its complement or a stable set of size 3,
Possible Partitions for $C_6$-Free Graphs

$C_6$ can be partitioned into two stable sets, a stable set and two cliques, and three cliques but not two cliques so $wpn(C_6)$ is 2.

If $S_1-Z$ is a clique then $S_{2-i}-Z$ must be $P_4$-free.

If $S_i-Z$ is a stable set then $S_{2-i}-Z$ must be the complement of a graph of girth 5.

Otherwise, since not both $S_1-Z$ and $S_2-Z$ contain a copy of $F$ if $F$ is $P_3$, its complement or a stable set of size 3, one is the disjoint union of cliques the other is complete multipartite
Possible Partitions for $C_6$-Free Graphs

$C_6$ can be partitioned into two stable sets, a stable set and two cliques, and three cliques but not two cliques so $\text{wpn}(C_6)$ is 2.

If $S_1$-$Z$ is a clique then $S_2$-$i$-$Z$ must be $P_4$-free. If $S_i$-$Z$ is a stable set then $S_2$-$i$-$Z$ must be the complement of a graph of girth 5.

Otherwise, since not both $S_1$-$Z$ and $S_2$-$Z$ contain a copy of $F$ if $F$ is $P_3$, its complement or a stable set of size 3, one is the disjoint union of cliques the other is complete multipartite and either the first has only two cliques or the second is the complement of a matching.
Typical $C_6$-Free Graphs

Almost every $C_6$-free graph can be partitioned into a stable set and the complement of a graph of girth 5.

Almost every $C_6$-free graph satisfies:

$$\chi(G) = O(\omega(G)^2).$$
Forbidding A Set of Cycles

If $\mathcal{F}$ is obtained by forbidding a set of cycles as induced subgraphs and the smallest cycle in this set has length $l$ then:

(a) if $l$ is neither 3 nor 6 then a.e. graph in $\mathcal{F}$ satisfies $\chi = \omega$,

(b) if $l$ is 6 then a.e. graph in $\mathcal{F}$ satisfies $\chi = O(\omega^2)$,

(c) If $l$ is 3 and we forbid no even cycle then almost every graph in $\mathcal{F}$ satisfies $\chi = \omega = 2$.

(d) If we forbid cycles of length 3 and 4 and allow only a finite number of cycle lengths then every graph in $\mathcal{F}$ satisfies $\chi = O(\omega)$.


(e) for other sets where $l$ is 3, the situation is less clear.
Forbidding A Set of Cycles

If $\mathcal{F}$ is obtained by forbidding a set of cycles as induced subgraphs and the smallest cycle in this set has length $l$ then:

(a) if $l$ is neither 3 nor 6 then a.e. graph in $\mathcal{F}$ satisfies $\chi = \omega$,

(b) if $l$ is 6 then a.e. graph in $\mathcal{F}$ satisfies $\chi = O(\omega^2)$,

(c) if $l$ is 3 and we forbid no even cycle then almost every graph in $\mathcal{F}$ satisfies $\chi = 2$.

(d) If we forbid cycles of length 3 and 4 and allow only a finite number of cycle lengths then every graph in $\mathcal{F}$ satisfies $\chi = O(\omega)$.

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Further Work

Disproving The Conjecture

Determining the structure of the graphs induced by the partition classes in typical H-free graphs for specific H and typical H.

Characterize completely the structure of typical graphs when we forbid a triangle and some other set of cycles, including at least one even one.
Thank you!